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Asymptotic Methods for Partial Differential Equations: The Reduced Wave Equation and Maxwell's Equations

ROBERT M. LEWIS and JOSEPH B. KELLER

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Contents

Introduction

Partial differential equations play a central role in many areas of physics, engineering, and applied mathematics. Existence and uniqueness theorems have been proved, and general properties of solutions have been studied, for large classes of problems for partial differential equations. However, explicit exact solutions of such problems, appropriate say for engineering applications, can be obtained only for relatively few problems; and often the analytical form even of these solutions is too complex to be useful for practical applications. Because of this, considerable effort has been devoted to the study of approximate solution methods. These fall mainly into two categories: numerical methods and asymptotic methods.

Because of the stimulus provided by the development of high speed digital computers numerical analysis has made tremendous strides in recent years, and for many problems involving partial differential equations, numerical methods are ideally suited. For some purposes, however, these methods are impractical or even useless. This is particularly true when one is primarily concerned with such questions as the functional dependence of the solution on the parameters and the data of the problem.

Asymptotic methods have been developed for some types of problems, in particular for certain problems involving a parameter. Such methods provide one or more terms of the asymptotic expansion (say for large values of the parameter) of the solution of the problem. They are applicable to many problems for which exact solutions are not available, and even for problems which have been solved exactly it often happens that only the asymptotic expansion of the solution is sufficiently simple to be useful in practical applications. Furthermore it is invariably true that the methods which yield the asymptotic expansion directly are very much simpler than the procedure which involves first finding the exact solution, and then its asymptotic expansion.

This report is devoted to the study of a certain class of asymptotic methods for linear partial differential equations. A central feature of such methods is the notion of "rays" which are curves or straight lines. The rays are of fundamental importance because all of the functions which make up the various terms of the asymptotic expansion can be shown to satisfy ordinary different equations along these curves. Thus, in a sense the method is one which reduces partial differential equations to ordinary differential equations. Often the latter can be solved to yield explicitly the desired asymptotic expansions. In some cases, however, the ordinary differential equations cannot be solved explicitly. This is a limitation of the method which is often overlooked.

The historical development of our subject is already suggested by the term "ray" which is a central idea in "geometrical optics." In our study of the reduced wave equation in chapter 4, we shall see that the geometrical optics solution of an optical problem is identical to the leading term of the asymptotic expansion (for large frequencies) of the solution of an appropriate problem for the reduced wave equation. Thus the asymptotic method shows that geometrical optics is a first approximation to "wave optics". But in addition to this important insight, the method provides further terms in the asymptotic expansion. These terms are of course particularly important in regions (such as "shadow regions") where the geometrical optics term is zero. The presence of small disturbances in geometrical shadow regions has been called the phenomenon of "diffraction". Thus we may say that the asymptotic theory of the reduced wave equation yields not only the classical geometrical optics, but also a new "geometrical theory of diffraction." We shall not attempt to summarize the history of either the classical or modern theory here. However, the references to this report provide an outline of many of the contributions which have constituted the modern development.

Let us indicate very briefly the steps involved in the asymptotic method. For problems which can be solved exactly, examination of the asymptotic expansion of the solution shows that it consists of a sum of terms, each of which is an asymptotic series involving a "phase function" and an infinite sequence of "amplitude functions". For complex problems, we therefore assume that the solution is also a sum of such series. By inserting such a series into the partial differential equation we find first that the phase function satisfies a first order partial differential equation which can be solved by the "method of characteristics." The characteristic curves are the rays which we have mentioned and the characteristic equations will be called "ray equations". Thus the phase function satisfies an ordinary differential equation along the rays. We also find that the amplitude functions satisfy ordinary differential equations along the rays. In order to find the rays and the phase and amplitude functions it is necessary to specify initial conditions for all of these ordinary differential equations. In some cases the initial conditions are a direct consequence of the data of the problem. In others the data are obtained from a "canonical problem." A canonical problem is a problem with the same local features as the given problem. It is, however, sufficiently simple to be solved exactly. The required initial conditions of the given problem are obtained by examination of the asymptotic expansion of the solution of the canonical problem. Our use of the word "simple" in connection with canonical problems is perhaps misleading. Often the solution of a canonical problem is an ambitious research project. Once it is found, however, (and expanded asymptotically) it yields the necessary information to complete the asymptotic solution of a great many problems which cannot be solved exactly. Thus the asymptotic method also provides a wide application for exact solutions of "simple" problems, hence an additional motivation for studying them.

It is by now clear that the asymptotic method involves several unproved assumptions. It is therefore reasonable to ask whether it can be proved that it does indeed yield the asymptotic expansion of the exact solution of the given problem. No general proofs of this fact have yet been given. Nevertheless there is abundant evidence of the validity of the method. The evidence is obtained either by comparison of the results of the asymptotic method with the asymptotic expansion of exact solutions (where such solutions are available), or by comparison with numerical and experimental methods.

Chapter A provides the most complete illustration of our method. In that chapter we have attempted to provide a unified summary of the existing literature on the asymptotic theory of the reduced wave equation. In chapter B we apply the same methods to Maxwell's equations. Most of the features there are the same as in chapter A, and the reader whose primary interest is in understanding the theory can omit chapter B. We have included it because we view the asymptotic method as a practical procedure for solving problems, and there is a large demand for the solution of problems involving the electromagnetic field. Furthermore the vector character of this field introduces complications which do not arise in the study of the reduced wave equation.

Throughout the report, vectors are denoted by capitals.

Acknowledgement

The authors wish to express their appreciation to the many members of the Courant Institute of Mathematical Sciences and others who participated in the 1963 summer seminar in asymptotic methods for partial differential equations at New York University. Their interest and enthusiasm provided an additional stimulus for the preparation of this report. They have also offered us valuable criticisms and comments, and some of the participants have indicated their intention of pursuing research projects suggested by the seminar.

Abstract

The asymptotic theory of the reduced wave equation and Maxwell's equations for high frequencies is presented. The theory is applied to representative problems involving reflection, transmission, and diffraction in homogeneous and inhomogeneous media. The report contains few new results. It is intended to unify and summarize the existing literature on the subject.

A. Asymptotic Methods for the Reduced Wave Equation

A1. Asymptotic Solution of the Reduced Wave Equation

Let us consider a real or complex function $v(t, X)$ which satisfies the wave equation

$$\Delta v - \frac{1}{c^2(X)} v_{tt} = 0. \tag{1}$$

Here the real valued function $c(X)$ is the propagation speed at the point X. We shall look for a product solution of (1) of the form $v = g(t)u(X)$. If we insert this form into (1) and separate variables, we obtain

$$c^2(X) \frac{\Delta u(X)}{u(X)} = \frac{g''(t)}{g(t)}. \tag{2}$$

Now we set $X = X_0$ in (2) and denote the (constant) value of the left-hand side by $-\omega^2$. Then (2) yields

$$-\omega^2 = g''(t)/g(t), \tag{3}$$

and on substituting (3) into (2) we obtain

$$\Delta u + \frac{\omega^2}{c^2(X)} u = 0. \tag{4}$$

Equation (4) for u is called the reduced wave equation or sometimes the Helmholtz equation. It is customary to introduce into it a constant reference speed c_0. In terms of c_0 we define the index of refraction $n(X) = c_0/c(X)$ and the propagation constant, or wave number, $k = \omega/c_0$. Then (4) becomes

$$\Delta u + k^2 n^2(X)\, u = 0. \tag{5}$$

The constant ω is called the angular frequency of the solution because two linearly-independent solutions of (5) are the periodic functions $g(t) = e^{-i\omega t}$ and $g(t) = e^{i\omega t}$. With them we can form the two linearly-independent product solutions $u(X)e^{-i\omega t}$ and $u(X)e^{i\omega t}$. Since the complex conjugate of every solution of (5) is also a solution of (5), it follows that every product solution of (1) of the form $u(X)e^{i\omega t}$ is the complex conjugate of a solution of the form

$$v(t,X) = u(X)e^{-i\omega t}. \tag{6}$$

Therefore it suffices to study solutions with "negative time factor" of the form (6). If a real solution v is required, the real part of (6) is such a solution.

We shall now consider the solution of (5) for large values of k. We begin with the observation that when $n(X)$ is constant, (5) admits the plane wave solutions

$$u(X,K) = s(K)e^{inK\cdot X}. \tag{7}$$

Here the propagation vector K is a real or complex vector of length $(K^2)^{1/2} = k$ and the amplitude $s(K)$ is a real or complex constant. In fact it follows from the fourier integral theorem that every solution of (5) with constant n is a superposition of plane wave solutions of the form (7). The exponential $e^{inK\cdot X}$ is called the phase factor of the solution and we shall call $nK\cdot X$ the phase. By analogy with (7) we shall seek solutions of (5) of the form

$$u(X) = s(X,k)e^{iks(X)}. \tag{8}$$

Upon inserting (8) into (5), and cancelling the phase factor e^{iks}, we obtain

$$k^2[(\nabla s)^2 - n^2]\, z + 2ik\nabla s\cdot\nabla z + ikz\Delta s + \Delta z = o. \tag{9}$$

To solve (9) for large values of k we assume that $z(X,k)$ can be expanded in inverse powers of k. It is convenient to write the expansion in terms of (ik) in the form

$$z(X,k) \sim \sum_{m=0}^{\infty} z_m(X)(ik)^{-m} = \sum z_m(X)(ik)^{-m},\; z_m = o \text{ for } m = -1, -2, \ldots. \tag{10}$$

We have used the sign of <u>asymptotic equality</u> in (10) to indicate that the series must be an <u>asymptotic expansion</u> of z as $k \to \infty$. This means that for each $n \geq 0$

$$z(X,k) = \sum_{m=0}^{n} z_m(X)(ik)^{-m} + o(k^{-n}). \tag{11}$$

By definition the order symbol denotes a term for which $\lim_{k\to\infty} k^n|o(k^{-n})| = o$. We will assume that the expansions of ∇z and Δz are obtained by termwise differentiation of (10). Upon inserting (10) into (9) we obtain

$$\sum_m (ik)^{1-m}\left\{[(\nabla s)^2 - n^2]z_{m+1} + [2\nabla s\cdot\nabla z_m + z_m\Delta s] + \Delta z_{m-1}\right\} \sim o. \tag{12}$$

From (12) it follows that the coefficient of each power of k must be zero. For $m = -1$ we obtain

$$[(\nabla s)^2 - n^2]z_o = o\,, \tag{13}$$

since $z_m \equiv o$ for $m = -1, -2, \ldots$. If, as we assume, $z_o \neq o$, (13) leads to the <u>eiconal equation</u> for s,

$$(\nabla s)^2 = n^2(X). \tag{14}$$

For $m = 0,1,2,\ldots$, the vanishing of the coefficients implies

$$2\nabla s\cdot\nabla z_0 + z_0\,\Delta s = 0 \tag{15}$$

and

$$2\nabla s\cdot\nabla z_m + z_m\,\Delta s = -\Delta z_{m-1}\,, \quad m = 1,2,\ldots \tag{16}$$

These equations are called the transport equations. We will see that z_0 can be obtained by solving (15) and the other z_m can be determined successively from (16).

A.2. Phase, wave-fronts and rays.

The eiconal equation (1.14) is a first order non-linear partial differential equation for $s(x)$. We could obtain required solutions of (1.14) by applying the general theory of first order partial differential equations*. However, the special form of (1.14) enables us to take a simplified (though equivalent) approach, and avoid some of the complications of the general theory.

The surfaces of constant phase, defined by $s(x)$ = constant, are called wavefronts. The curves orthogonal to them can be used to solve (1.14) for $s(X)$. These curves are called rays. (In the general theory they are called the characteristic curves.) The equation of a ray may be written in terms of a parameter σ in the form

$$X = (x_1, x_2, x_3) = X(\sigma). \tag{1}$$

The condition of orthogonality is

$$\frac{dx_j}{d\sigma} = \lambda s_{x_j}; \quad j = 1,2,3. \tag{2}$$

Here $\lambda(X)$ is an arbitrary proportionality factor. Upon dividing (2) by λ and differentiating with respect to σ we obtain

$$\frac{d}{d\sigma}\left(\frac{1}{\lambda}\frac{dx_j}{d\sigma}\right) = \frac{d}{d\sigma} s_{x_j} = \sum_{i=1}^{3} s_{x_j x_i}\frac{dx_i}{d\sigma} = \lambda \sum_{i=1}^{3} s_{x_j x_i} s_{x_i} = \frac{\lambda}{2}\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{3} s_{x_i}^2\right);$$

$$j = 1,2,3. \tag{3}$$

Now (3) and (1.14) yield

$$\frac{1}{\lambda}\frac{d}{d\sigma}\left(\frac{1}{\lambda}\frac{dx_j}{d\sigma}\right) = \frac{\partial}{\partial x_j}\left(\frac{n^2}{2}\right); \quad j = 1,2,3. \tag{4}$$

In addition (2) and (1.14) give

$$\sum_{j=1}^{3}\left(\frac{dx_j}{d\sigma}\right)^2 = \lambda^2 n^2. \tag{5}$$

*See [5], Chapter 2.

Equations (4) are a system of three second-order ordinary differential equations for the rays $X(\sigma)$, and (5) determines the variation of the parameter σ along a ray, once λ has been chosen. We call these equations the <u>ray equations</u>. It is to be noted that s does not occur in them. Hence the rays are determined solely by $n(X)$, once initial values for (4) are specified. Of all the rays, a two parameter family are orthogonal to the wave-fronts of a given phase function s.

If we choose $\lambda = n^{-1}$ the ray equations take the form

$$n \frac{d}{d\sigma}\left(n \frac{dx_j}{d\sigma}\right) = \frac{\partial}{\partial x_j}\left(\frac{n^2}{2}\right); \quad j = 1,2,3; \tag{6}$$

$$\sum_{j=1}^{3}\left(\frac{dx_j}{d\sigma}\right)^2 = 1. \tag{7}$$

From (7) we see that for this choice of λ, σ is just arc-length along the ray. If we choose $\lambda = 1$ and denote σ by τ the ray equations take the simple form

$$\frac{d^2x_j}{d\tau^2} = \frac{\partial}{\partial x_j}\left(\frac{n^2}{2}\right); \quad j = 1,2,3; \tag{8}$$

$$\sum_{j=1}^{3}\left(\frac{dx_j}{d\tau}\right)^2 = n^2 \tag{9}$$

From (7) and (9) it is clear that if σ denotes arc-length

$$d\sigma = \sqrt{\sum_j (dx_j)^2} = n\,d\tau. \tag{10}$$

To solve the eiconal equation (1.14) for s we note that (1.14) and (2) yield, for the derivative of s along a ray, the result

$$\frac{d}{d\sigma} s\,[X(\sigma)] = \nabla s \cdot \frac{dX}{d\sigma} = \lambda(\nabla s)^2 = \lambda n^2. \tag{11}$$

Upon integrating (11) with respect to σ we obtain

$$s[X(\sigma)] = s[X(\sigma_0)] + \int_{\sigma_0}^{\sigma} \lambda[X(\sigma')]n^2[X(\sigma')]d\sigma' \,. \tag{12}$$

When $\lambda = n^{-1}$, σ denotes arc-length and (12) becomes

$$s(\sigma) = s(\sigma_0) + \int_{\sigma_0}^{\sigma} n(\sigma')\, d\sigma'. \tag{13}$$

Here we have written $s(\sigma)$ for $s[X(\sigma)]$ and used a similar notation for n.

Similarly when $\lambda = 1$, (12) becomes

$$s(\tau) = s(\tau_0) + \int_{\tau_0}^{\tau} n^2(\tau')d\tau'. \tag{14}$$

(13) and (14) provide simple formulas for the value of s at any point on a ray in terms of the value at a given point.

A3. Solution of the transport equations for the amplitudes

In the preceding section the rays were used to obtain the solution $s(X)$ of the eiconal equation (1.14). They can also be used to solve the transport equations (1.15) and (1.16). We first note that $\nabla s \cdot \nabla z_m$ is proportional to the directional derivative of z_m in the direction of ∇s, which is just the ray direction. In fact from (2.2) we obtain

$$\nabla s \cdot \nabla z_m = \frac{1}{\lambda}\frac{dX}{d\sigma} \cdot \nabla z_m = \frac{1}{\lambda}\frac{d}{d\sigma}\; z_m[X(\sigma)]. \tag{1}$$

Thus we see that the transport equations (1.15), (1.16) are, in fact, first order ordinary differential equations along the rays, and may be written as

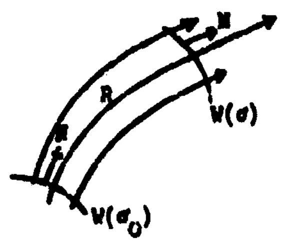
$$\frac{2}{\lambda}\frac{dz_0}{d\sigma} + z_0\,\Delta s = 0, \tag{2}$$

$$\frac{2}{\lambda}\frac{dz_m}{d\sigma} + z_m\Delta s = -\Delta z_{m-1}; \quad m = 1,2,\ldots. \tag{3}$$

We will first obtain the solution z_0 of the homogeneous equation (2) and then use it to obtain the solution of the inhomogeneous equation (3) by standard methods. Actually, the solution of (2) is most easily obtained by returning to the form (1.13) and noting that that equation implies

$$\nabla\cdot(z_0^2\,\nabla s) = z_0(2\nabla z_0\cdot\nabla s + z_0\Delta s) = 0. \tag{4}$$

Given a ray, we now consider a region R of X-space bounded by a tube of rays containing the given ray, and two segments of wave-fronts $W(\sigma_0)$ and $W(\sigma)$ at the points σ_0 and σ of the given ray (see figure).



Thus ∇s is parallel to the sides of the tube and normal to its ends. We now apply Gauss' theorem to the region R. By virtue of (4) we obtain

$$0 = \iiint_R \nabla\cdot(z_0^2\,\nabla s)\,dX = \iint_{W(\sigma)} z_0^2\,\nabla s\cdot N\,da - \iint_{W(\sigma_0)} z_0^2\,\nabla s\cdot N\,da. \tag{5}$$

Here N is a unit vector orthogonal to the wave-fronts. However from (1.14) we see that $\nabla s\cdot N = n$. Therefore by shrinking the tube of rays to the given ray we obtain

$$z_0^2(\sigma)n(\sigma)da(\sigma) = z_0^2(\sigma_0)n(\sigma_0)da(\sigma_0). \tag{6}$$

Let us now choose an arbitrary point σ_1 on the ray and set

$$\xi(\sigma) = \frac{da(\sigma)}{da(\sigma_1)}. \tag{7}$$

$\xi(\sigma)$ is called the expansion ratio since it measures the expansion of a tube of rays. It is just the Jacobian of the mapping by rays of $W(\sigma)$ on $W(\sigma_1)$. From (6) and (7) we now obtain the solution of (2) in the form

$$z_0(\sigma) = z_0(\sigma_0)\left[\frac{\xi(\sigma_0)n(\sigma_0)}{\xi(\sigma)\,n(\sigma)}\right]^{1/2} . \tag{8}$$

From (8) we see that $z_0(\sigma)$ varies inversely as the square root of $n\xi$ along a ray, so that when ξ diminishes z_0 increases. Thus convergence of the rays tends to increase z_0 and divergence of them tends to decrease it. The physical interpretation is perhaps more clearly seen in (6) which states that the energy flux $z_0^2 n da$ is constant along an infinitesimal tube of rays.

In order to obtain the solution of the inhomogeneous equation (3) we introduce the solution

$$r(\sigma) = \left[\frac{\xi(\sigma_0)n(\sigma_0)}{\xi(\sigma)n(\sigma)}\right]^{1/2} \tag{9}$$

of the homogeneous equation and note that $r(\sigma_0) = 1$. Then by the method of "variation of parameters" we look for a function $w(\sigma)$ such that

$$z_m(\sigma) = w(\sigma)r(\sigma). \tag{10}$$

If we differentiate (10) with respect to σ, insert in (3), and note that r satisfies (2), we obtain

$$\frac{dw}{d\sigma} = -\frac{\lambda}{2r}\Delta z_{m-1}. \tag{11}$$

It follows that, up to an arbitrary additive constant, w is given by

$$w(\sigma) = -\frac{1}{2}\int_{\sigma_0}^{\sigma}\frac{\lambda}{r}\Delta z_{m-1}\,d\sigma' \tag{12}$$

and the general solution of (3) is

$$z_m(\sigma) = c_1 r(\sigma) + r(\tau)w(\sigma) = c_1 r(\sigma) - \frac{1}{2}\int_{\sigma_0}^{\sigma} \frac{r(\sigma)}{r(\sigma')}\lambda(\sigma')\Delta z_{m-1}(\sigma')d\sigma'. \qquad (13)$$

By setting $\sigma = \sigma_0$ we see that $c_1 = z_m(\sigma_0)$, hence

$$z_m(\sigma) = z_m(\sigma_0)\left[\frac{\xi(\sigma_0)n(\sigma_0)}{\xi(\sigma)n(\sigma)}\right]^{1/2} - \frac{1}{2}\int_{\sigma_0}^{\sigma}\left[\frac{\xi(\sigma')n(\sigma')}{\xi(\sigma)n(\sigma)}\right]^{1/2}\Delta z_{m-1}(\sigma')\lambda(\sigma')d\sigma';$$

$$m = 1,2,\ldots . \qquad (14)$$

If we choose λ to be n^{-1} then σ denotes arc-length along the ray and $\lambda(\sigma')$ must be replaced by $n^{-1}(\sigma')$ in (14). If we choose λ to be 1 then (14) becomes

$$z_m(\tau) = z_m(\tau_0)\left[\frac{\xi(\tau_0)n(\tau_0)}{\xi(\tau)n(\tau)}\right]^{1/2} - \frac{1}{2}\int_{\tau_0}^{\tau}\left[\frac{\xi(\tau')n(\tau')}{\xi(\tau)n(\tau)}\right]^{1/2}\Delta z_{m-1}(\tau')d\tau';$$

$$m = 1,2,\ldots . \qquad (15)$$

A4. <u>The case of homogeneous media</u>.

The solution $v(t,X)$ of (1.1) represents a disturbance in a physical medium which is charcterized by the propagation speed $c(X)$ or the index of refraction $n(X) = c_0/c(X)$. The medium will be called <u>homogeneous</u> if these functions are constant. In this case our earlier results simplify considerably.

First we see, from (2.8), that the rays are straight lines, and from (2.1?) that

$$s(\sigma) = s(\sigma_0) + n(\sigma-\sigma_0). \qquad (1)$$

Here σ denotes arc-length along a ray. If σ is measured on all rays from a wave-front $s(X) = s_0$ then

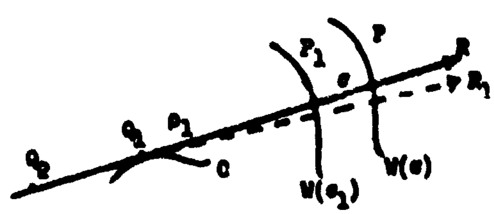
$$s(\sigma) = s(0) + n\sigma = s_0 + n\sigma. \qquad (2)$$

Hence the distance from the wave-front $s(X) = s_0$ to the wave-front $s(X) = s_1$ is just $(s_1 - s_0)/n$. But this distance is the same on every ray. Therefore the wave-fronts form a family of parallel surfaces.

The expressions (3.8) and (3.14) for s_0 and s_m can now be considerably simplified by appealing to some elementary facts of the differential geometry of surfaces: Let P_1 be a regular point on the surface S, and let N be the unit normal vector to S at P_1. Every plane through P_1 which is parallel to N cuts S in a curve called a <u>normal section</u>. Let κ denote the curvature and $\rho = \kappa^{-1}$ the radius of curvature of the normal section at the point P_1. Then κ depends on the direction of the plane. It can be shown that there exist two orthogonal directions, the <u>principal directions</u> at P_1, for which κ has maximum and minimum values. These values are called the <u>principal curvatures</u> and will be denoted by κ_1 and κ_2. Their product

$$G = \kappa_1 \kappa_2 = \frac{1}{\rho_1 \rho_2} \tag{3}$$

is called the <u>Gaussian curvature</u> of S at P_1. Let us now take S to be the wave-front $W(\sigma_1)$ (see section A3) and let P_1 be the point of intersection of the ray R of interest and the surface $W(\sigma_1)$. The plane of the following figure is chosen to be the plane which cuts $W(\sigma_1)$ in the normal section whose radius of curvature is ρ_1.



The ray R intersects the (parallel) wave-front $W(\sigma)$ at a point, P. Without loss of generality we may measure σ from the wave-front $W(\sigma_1)$. Then $\sigma_1 = 0$ and the distance from P_1 to P is σ. Since the wave-fronts are parallel the plane of the figure cuts $W(\sigma)$ in a normal section with radius of curvature $\rho_1 + \sigma$, and the plane through R orthogonal to the plane of the figure cuts $W(\sigma)$ in a normal section with radius of curvature $\rho_2 + \sigma$. Furthermore it is clear that $\rho_1 + \sigma$ and $\rho_2 + \sigma$ are the principal radii of curvature of $W(\sigma)$ at P.

Q_1 is the center of curvature corresponding to ρ_1. At that point the ray R and its neighboring ray R_1, an infinitesimal distance away, intersect. More precisely Q_1 is a point on an envelope of the family of rays. There is a similar point Q_2 corresponding to the other principal radius of curvature ρ_2, and the two points Q_1 and Q_2 lie on a two-sheeted envelope of the ray family. This surface, C, is called the caustic or caustic surface of the ray family, and the rays are tangent to it. Sometimes the caustic degenerates to a curve or a point. In the latter case it is called a focus. The family of rays itself (each normal to $W(\sigma_1)$, hence to all the wave fronts) is called a normal congruence of rays.

Now let $d\theta_1$ be the angle between the rays R and R_1. This angle subtends arcs on $W(\sigma_1)$ and $W(\sigma)$ whose lengths are $\rho_1 d\theta_1$ and $(\rho_1+\sigma)d\theta_1$ respectively. Similarly we may consider a ray R_2 in the plane normal to the plane of the figure which makes an angle $d\theta_2$ with R. The latter angle subtends arcs on $W(\sigma_1)$ and $W(\sigma)$ whose lengths are $\rho_2 d\theta_2$ and $(\rho_2+\sigma)d\theta_2$ respectively. It follows now (see section A3) that the expansion ratio is given by

$$\xi(\sigma) = \frac{da(\sigma)}{da(\sigma_1)} = \frac{(\rho_1+\sigma)d\theta_1(\rho_2+\sigma)d\theta_2}{\rho_1 d\theta_1\ \rho_2 d\theta_2} = \frac{(\rho_1+\sigma)(\rho_2+\sigma)}{\rho_1\rho_2} = \frac{g(0)}{g(\sigma)} . \tag{4}$$

Here $g(\sigma)$ is the Gaussian curvature of $W(\sigma)$ at P.

Equation (4) enables us to write the equations (3.8) and (3.14) for the amplitude function z_0 and z_m in the simple form

$$z_0(\sigma) = z_0(\sigma_0)\left[\frac{(\rho_1+\sigma_0)(\rho_2+\sigma_0)}{(\rho_1+\sigma)(\rho_2+\sigma)}\right]^{1/2} \tag{5}$$

$$z_m(\sigma) = z_m(\sigma_0)\left[\frac{(\rho_1+\sigma_0)(\rho_2+\sigma_0)}{(\rho_1+\sigma)(\rho_2+\sigma)}\right]^{1/2} - \frac{1}{2n}\int_{\sigma_0}^{\sigma}\left[\frac{(\rho_1+\sigma')(\rho_2+\sigma')}{(\rho_1+\sigma)(\rho_2+\sigma)}\right]^{1/2}\Delta z_{m-1}(\sigma')d\sigma'; \quad m = 1,2,\ldots . \tag{6}$$

ρ_1 and ρ_2 are the principal radii of curvature of the wave-front at $\sigma = 0$.

At this point it might be well to point out the connection between our subject and the subject of <u>geometrical optics</u>, for it is clear that many of the terms we have introduced such as "ray", "wave front", "caustic", "focus", have been borrowed from that subject. A closer comparison shows that geometrical optics consists of a set of rules for the construction of a function to represent wave phenomena. This function turns out to be identical to the function $e^{iks(X)}z_0(X)$ which is the leading term of our asymptotic expansion. (Our remarks here apply as well to the case of inhomogeneous media, $n(X) \neq$ const., as to the case of homogeneous media). Indeed the asymptotic theory explains the apparent paradox that two quite different basic theories, geometrical optics and wave optics (i.e. the wave equation), have been used successfully to describe the same physical phenomena. Of course, the more restricted theory, geometrical optics, can be expected to be valid only at high frequencies (large k). The leading term of our expansion is often called, appropriately, the <u>geometrical optics term</u>. The higher terms supply corrections to geometrical optics. In recent years the subject of geometrical optics has been successfully generalized, principally by J. B. Keller and his co-workers, to explain the phenomena of diffraction and other phenomena not accounted for by the classical theory. We will have more to say later about this geometrical theory of diffraction.

Let us also briefly examine the connection between our subject and the exact theory of the reduced wave equation. In special cases, i.e. for special choices of the function $n(X)$, and for problems involving boundaries with special geometrical features, problems for the reduced wave equation can be solved exactly by the method of separation of variables. When such exact solutions are expanded asymptotically for large k, the expansions are found to agree exactly with the expansions we are constructing. However the class of problems which can be solved exactly is extremely restricted and even for this class of problems the asymptotic solution can be obtained much more quickly and easily by the present methods. Of course the following question remains: For a very wide class of problems the solution is known to exist and to be unique, but explicit construction of the exact solution is not possible. In these cases is our "asymptotic expansion" indeed the asymptotic expansion of the exact solution? At present no general proofs can be given to answer this question. Nevertheless considerable experience and comparison with exact solutions which can be obtained provides us with confidence that the answer to the question is affirmative.

A5. Waves

Let us summarize our results for the asymptotic solution $u(X)$ of the reduced wave equation (1.5). From (1.9) and (1.10) we have

$$u(X) \sim e^{iks(X)} \sum_{m=0}^{\infty} z_m(X)\,(ik)^{-m}. \tag{1}$$

The phase function $s(X)$ is a solution of the eiconal equation, and according to (2.13) is given, at the point $X(\sigma)$ on a ray, by

$$s[X(\sigma)] = s[X(\sigma_0)] + \int_{\sigma_0}^{\sigma} n[X(\sigma')]\,d\sigma'. \tag{2}$$

Here σ denotes arc-length along a ray. The rays are determined by the ray equations (2.6). The amplitude $z_0(X)$ is given at the point $X(\sigma)$ on a ray by

$$z_0[X(\sigma)] = z_0[X(\sigma_0)]\left[\frac{\xi(\sigma_0)\,n[X(\sigma_0)]}{\xi(\sigma)\,n[X(\sigma)]}\right]^{\frac{1}{2}}, \tag{3}$$

and the other $z_m(X)$ are given recursively by (3.14).

When the functions $s(X)$ and $z_m(X)$ have been determined, the series (1) is an asymptotic solution of the reduced wave equation. Such a solution will be called a <u>wave</u>. It frequently happens that more than one ray associated with a wave passes through a given point X. In such cases the value of the wave at X is given by a sum of expressions of the form (1), one for each ray passing through X. If no waves pass through a point the value of the wave at that point is zero. Since the reduced wave equation is linear, the sum of any number of solutions of it is also a solution. We shall see that the asymptotic solution of a given problem for the reduced wave equation will, in general, consist of a sum of waves, appropriately selected to satisfy the data of the problem.

In order to determine a wave uniquely, initial values for s must first be prescribed. These initial values determine the family of rays associated with the wave and can be used in (2) to determine s at every point on every ray. In addition, initial values of the functions $z_m(X)$ must be prescribed at one point on each ray. Then these functions are given at every point on every ray by (3) and (3.14). The initial values for s and the z_m are determined by the data of the problem. For example in a <u>radiation</u> problem, in which the solution is generated by a source, the source will determine which rays occur and what the initial values are on them. In a boundary value problem the inhomogeneous boundary data will determine the initial values.

To recapitulate: A solution consists of a sum of waves. A wave is uniquely determined by prescribing initial values for $s(X)$ and initial values for the $z_m(X)$ on each ray. Thus the first step in determining a wave is to solve the initial value problem for s. This is the subject of the next section.

A6. <u>The initial value problem for the eiconal equation</u>.

In the usual treatment of the initial value problem for a first order partial differential equation, initial values of the solution are prescribed on a surface. In our treatment of the eiconal equation we will also have to consider lower dimensional initial manifolds, specifically curves and points. We will consider these initial value problems in the order of increasing dimension of the initial manifold, i.e., point, curve, and surface. For none of these problems is the solution $s(X)$ uniquely determined by the value of s on the initial manifold. However, the solution which we require for our construction of a wave is uniquely determined by the additional condition that it be "outgoing" from the initial manifold.

A solution $s(X)$ of the eiconal equation will be said to be <u>outgoing</u> with respect to a manifold M if at M the normal derivative of s, $\nabla s \cdot N$, is positive for every outward normal N to M. If M is a point, then every direction from M is normal; if M is a curve, the normal directions at a point lie in the plane orthogonal to the curve; and if M is a surface there are two normal directions at every point, one on each side of the surface.

The outgoing condition is consistent with the physical picture of a disturbance spreading out from a source (whether that source be a primary one or, as in the case of reflection by surfaces and diffraction by curves and points, a secondary one). Mathematically, the "outgoing condition" is the asymptotic analogue of the radiation condition, without which the exact solution of a problem for the reduced wave equation is not uniquely determined.

For the initial value problem with a point initial manifold P, we require an outgoing solution $s(X)$ which satisfies the condition

$$s(P) = s_o . \tag{1}$$

Clearly the solution is obtained by finding all the rays that emanate from P. Then, on each ray, $s(X)$ is given by

$$s[X(\sigma)] = s_o + \int_o^\sigma n\,[X(\sigma')]d\sigma'. \tag{2}$$

When the source is a curve C we may describe it parametrically by the equation $X = X_o(\eta)$ where η denotes arclength along C. Let the prescribed value of s on C be $s[X_o(\eta)] = s_o(\eta)$. Differentiating this equation with respect to η yields

$$\nabla s \cdot \frac{dX_o}{d\eta} = \frac{ds_o}{d\eta} . \tag{3}$$

Let us introduce the angle $\theta(\eta)$ between ∇s and the unit tangent vector $\frac{dX_o}{d\eta}$ to the curve C at the point η. Then since the length of ∇s is n

$$\cos\theta(\eta) = \frac{1}{n[X_o(\eta)]}\,\frac{ds_o}{d\eta} . \tag{4}$$

Because the direction of ∇s is the ray direction, $\beta(\eta)$ is just the angle between a ray leaving the curve C at η and the tangent to C at η. The rays are those which emanate from the curve C, at every point along it, making the angle β with the tangent to C at the point. β is given by (4). Thus the initial directions of the rays emanating from each point η on C lie on a cone, the tangent to C at η being the axis of the cone and $\beta(\eta)$ being the semi-angle of the cone. Then, on every ray $X(\sigma;\eta)$ lying on the conoid emanating from the point $X_0(\eta)$, s is given by

$$s[X(\sigma,\eta)] = s_0(\eta) + \int_0^\sigma n[X(\sigma', \eta)]\, d\sigma' . \tag{5}$$

In the special case in which $\frac{ds_0}{d\eta} = 0$, (4) shows that $\beta(\eta) = \pi/2$ so the cone is a plane normal to C.

When the initial manifold is a surface, S, we may write its equation parametrically as $X = X_0(\eta_1,\eta_2)$. It is convenient to choose the parameters η_1 and η_2 to be arclengths along orthogonal curves on S. Let the prescribed value of s on S be $s[X_0(\eta_1,\eta_2)] = s_0(\eta_1,\eta_2)$. Differentiation of this equation with respect to η_1 and η_2 yields

$$\nabla s \cdot \frac{\partial X_0}{\partial \eta_j} = \frac{\partial s_0}{\partial \eta_j} \; ; \; j = 1, 2. \tag{6}$$

At a point P on S, $\frac{\partial X_0}{\partial \eta_1}$ and $\frac{\partial X_0}{\partial \eta_2}$ are orthogonal unit vectors lying in the tangent plane to S at P. Let β_j denote the angle between ∇s and $\frac{\partial X_0}{\partial \eta_j}$. Then (6) yields, at the point P,

$$\cos \beta_j = \frac{1}{n} \frac{\partial s_0}{\partial \eta_j} \; ; \; j = 1,2. \tag{7}$$

These equations determine two directions at P on opposite sides of S. These are the possible directions of ∇s. Thus two rays emanate from each point of S on opposite sides of the surface. On each of the two rays $X(\sigma;\eta_1,\eta_2)$ emanating from the point $X_0(\eta_1,\eta_2)$ on S, s is given by

$$s[X(\sigma;\eta_1,\eta_2)] = s_0(\eta_1,\eta_2) + \int_0^\sigma n[X(\sigma';\eta_1,\eta_2)]\, d\sigma' . \tag{8}$$

For completeness we also mention the characteristic initial value problem for the eiconal equation: We assume that the initial values s_o satisfy the "surface eiconal equation": *

$$(\nabla s_o)^2 = n^2 \qquad \text{on } S \ . \tag{9}$$

Then if we choose as surface co-ordinate curves the level curves s_o = const. and the orthogonal gradient curves, we may introduce the surface parameters $\tau_1 = s_o$ and τ_2. τ_2 is some parameter that labels the gradient curves, e.g., arc-length along one level curve. If η_1 and η_2 denote arc-length parameters corresponding to τ_1 and τ_2 then (9) implies

$$\frac{\partial s_o}{\partial \eta_1} = \frac{\partial \tau_1}{\partial \eta_1} = n \tag{10}$$

and clearly

$$\frac{\partial s_o}{\partial \eta_2} = \frac{\partial s_o}{\partial \tau_2} \frac{d\tau_2}{d\eta_2} = 0. \tag{11}$$

It follows from (7) that $\beta_1 = 0$ and $\beta_2 = \pi/2$, i.e., the rays are tangent to S. In this case since S is everywhere tangent to rays (i.e., characteristics) S is said to be a characteristic surface. One outgoing tangent ray $X(\sigma;\tau_1,\tau_2)$ emanates from every point $X_o(\tau_1,\tau_2)$ on S. On this ray s is given by

$$s[X(\sigma;\tau_1,\tau_2)] = \tau_1 + \int_0^\sigma n[X(\sigma';\tau_1,\tau_2)]d\sigma' \ . \tag{12}$$

The surface gradient curves τ_2 = const. are everywhere tangent to rays and may be called surface rays. They play a central role in our later discussion of diffraction by smooth bodies.

A7. Radiation from sources

One way of characterizing a source is by giving the values of the phase function $s(X)$, as well as the amplitude coefficients $z_m(X)$, at every point of the source manifold. Usually, however, the source is characterized in some other way and then the values of s and z_m must be derived by procedures which we will discuss

* The surface gradient ∇ is defined in Section I7.

shortly. Let us now suppose that these values are given and examine the construction of the resulting wave.

The phase function $s(X)$ as well as the rays are determined by the procedures of the preceding section. The amplitude coefficients may be obtained by means of (3.14) which we rewrite in the form

$$z_m(\sigma) = z_m(\sigma_o)\left[\frac{da(\sigma_o)n(\sigma_o)}{da(\sigma)n(\sigma)}\right]^{\frac{1}{2}} - \frac{i}{2}\int_{\sigma_o}^{\sigma}\left[\frac{da(\sigma')n(\sigma')}{da(\sigma)n(\sigma)}\right]^{\frac{1}{2}}\frac{\Delta z_{m-1}(\sigma')}{n(\sigma')}\,d\sigma'; \qquad (1)$$

$$m = 0,1,2,\dots .$$

Here σ denotes arclength and $z_{-1}(X) \equiv 0$.

If the source manifold is a surface S, then on every outgoing ray we may measure σ from S and $z_m(\sigma)$ is given by (1) with σ_o replaced by 0. We are assuming that $z_m(0)$ is given.

For point and line sources, the source manifold is a caustic of the resulting ray system and hence the formulas for the functions z_m become infinite at the source. In these cases the source values of the z_m may be characterized by appropriate limiting conditions. These conditions will be given only for the case m=0.*

For a point source, let $d\Omega$ be an element of solid angle of the starting directions of the rays. Then for sufficiently small σ_o, $da(\sigma_o) \sim \sigma_o^2 d\Omega$. If we introduce this expression in (1) and let $\sigma_o \to 0$ we obtain

$$z(\sigma) = \tilde{z}(0)\left[\frac{d\Omega}{da(\sigma)}\;\frac{n(0)}{n(\sigma)}\right]^{\frac{1}{2}} . \qquad (2)$$

We have omitted the subscript "o". In (2),

$$\tilde{z}(0) = \lim_{\sigma_o \to 0} \sigma_o z(\sigma_o). \qquad (3)$$

We will assume that for a point source, $\tilde{z}(0)$ is given. For the case of homogeneous media, n is constant and $da(\sigma) = \sigma^2 d\Omega$. Then (2) becomes

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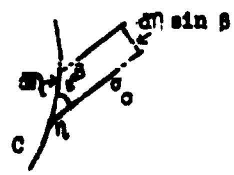
*The analogous formulas for m=1,2,... are more complicated and will not be required in the sequel.

$$z(\sigma) = \frac{\tilde{z}(0)}{\sigma} \quad . \tag{4}$$

For the source distributed on the curve C (see section A6) it is clear from the following figure that for sufficiently small σ_o,

$$da(\sigma_o) \sim d\eta \sin\beta \, \sigma_o d\theta. \tag{5}$$



Here $d\theta$ is an element of angle between two rays lying on the cone of rays which emanate from the point η of C. If we introduce (5) in (1) and let $\sigma_o \to 0$, we obtain

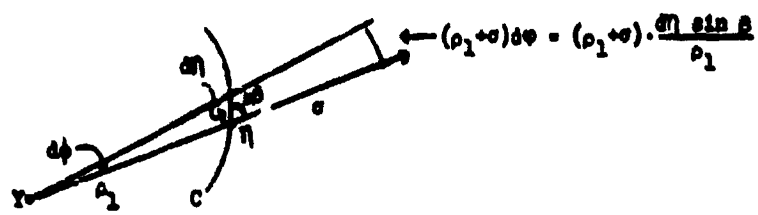
$$z(\sigma) = \tilde{z}(0)\left[\frac{d\eta d\theta n(0)}{da(\sigma)n(\sigma)} \sin\beta\right]^{\frac{1}{2}} \quad . \tag{6}$$

Here

$$\tilde{z}\,(0) = \lim_{\sigma_o\to 0} \sigma_o^{\frac{1}{2}}\, z\,(\sigma_o) \ . \tag{7}$$

Again we assume that $\tilde{z}\,(0)$ is given. For the case of homogeneous media we can see from the following figure that

$$da(\sigma) = \sigma d\theta \cdot (\rho_1 + \sigma)d\varphi = \sigma d\theta(1 + \frac{\sigma}{\rho_1})\sin\beta \ d\eta. \tag{8}$$



Here ρ_1 is the distance between the two caustic points on the ray emanating from the point η on C. The point η itself is one caustic point. The other point Y may (as in the figure) lie on the backward extension of the ray.

If we introduce (8) in (6) that equation becomes

$$z(\sigma) = \left[\sigma\left(1 + \frac{\sigma}{\rho_1}\right)\right]^{-\frac{1}{2}} z(0). \tag{9}$$

$-\rho_1$ is the signed distance from the curve to the other caustic along the ray in the ray direction (i.e., the direction of increasing σ). This distance can be found by deriving the equation of the caustic: A variable point Y on the cone of rays emanating from the point $X_o(\eta)$ satisfies the equation

$$(Y-X_o)\cdot\dot{X}_o = |Y-X_o| \cos\beta. \tag{10}$$

Here the dot denotes differentiation with respect to η, the arclength parameter on C. Differentiation of (10) with respect to η yields

$$(Y-X_o)\cdot\ddot{X}_o - 1 = -|Y-X_o|\dot{\beta}\sin\beta - \frac{Y-X_o}{|Y-X_o|}\cdot\dot{X}_o\cos\beta. \tag{11}$$

By inserting (10) in (11) we obtain

$$(Y-X_o)\cdot\ddot{X}_o = 1 - \dot{\beta}\sin\beta\,\frac{(Y-X_o)\cdot\dot{X}_o}{\cos\beta} - \cos^2\beta. \tag{12}$$

or

$$(Y-X_o)\cdot(\ddot{X}_o + \dot{\beta}\tan\beta\,\dot{X}_o) = \sin^2\beta. \tag{13}$$

We now introduce the unit tangent vector $T = \dot{X}_o$ and the unit normal vector $N = \rho\ddot{X}_o$ to the curve C at the point $X_o(\eta)$. ρ denotes the radius of curvature of the curve at that point. Then, from (13) and (10), the caustic is given by the two equations

$$(Y-X_o)\cdot(N+\rho\,\dot{\beta}\tan\beta\,T) = \rho\sin^2\beta \tag{14}$$

$$(Y-X_o)\cdot T = |Y-X_o|\cos\beta. \tag{15}$$

Eliminating η from (14) and (15) would yield a single equation for the caustic surface.

Now let δ be the angle between the ray and the vector N. Then if Y is the caustic point on the ray

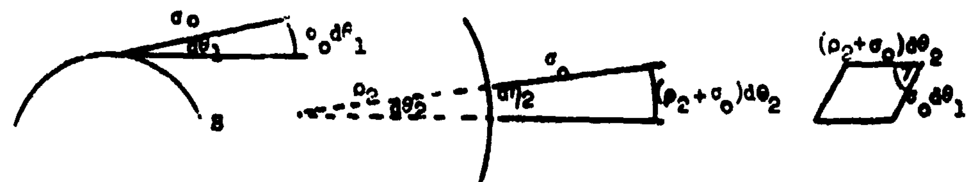
$$(Y-X_o)\cdot N = -\rho_1 \cos \delta, \quad (Y-X_o)\cdot T = -\rho_1 \cos \beta \tag{16}$$

and (14) yields

$$\rho_1 = -\frac{\rho \sin^2 \beta}{\rho \dot{\beta} \sin \beta + \cos \delta} . \tag{17}$$

In (17) the direction in which arclength increases along C is immaterial since both $\sin \beta$ and $\dot{\beta}$ are unchanged when η is replaced by $-\eta$.

We conclude this section with a discussion of radiation from a characteristic surface S (see the last paragraph of section A6). In this case the rays are tangent to S and S is a caustic of the ray system. The following figure shows the "side," "top," and "end" views of an infinitesimal tube of rays leaving the surface S:



(For sufficiently small σ_o the rays are approximately straight.) Here θ_2 (η_2) measures the angle at which rays emanate from the curve s_o = const. and

$$\rho_2 = \frac{d\eta_2}{d\theta_2} .$$

From the figure we see that for sufficiently small σ_o

$$da(\sigma_o) \sim \sigma_o(\rho_2+\sigma_o) \sin \gamma \, d\theta_1 d\theta_2 . \tag{18}$$

If we introduce (18) in (1) and let $\sigma_o \to 0$ we obtain

$$z(\sigma) = \tilde{z}(0) \left[\rho_2 \sin \gamma \frac{d\theta_1 d\theta_2}{da(\sigma)} \frac{n(0)}{n(\sigma)}\right]^{\frac{1}{2}} . \tag{19}$$

Here

$$\tilde{z}(0) = \lim_{\sigma_o \to 0} \sigma_o^{\frac{1}{2}} z(\sigma_o). \tag{20}$$

For the case of homogeneous media

$$da(\sigma) = \sigma(\rho_2 + \sigma) \sin\gamma \, d\theta_1 d\theta_2 \tag{21}$$

and (19) becomes

$$z(\sigma) = \left[\sigma(1 + \frac{\sigma}{\rho_2})\right]^{-\frac{1}{2}} Z(0). \tag{22}$$

For some purposes (e.g., where S is a plane) (19) is inconvenient. It can be replaced by using (1) and (20). We define

$$\frac{d\mathcal{a}(0)}{da(\sigma)} = \lim_{\sigma_0 \to 0} \frac{da(\sigma_0)}{\sigma_0 da(\sigma)} . \tag{23}$$

Then

$$z(\sigma) = Z(0)\left[\frac{d\mathcal{a}(0)}{da(\sigma)} \frac{n(0)}{n(\sigma)}\right]^{\frac{1}{2}} . \tag{24}$$

A8. Isotropic point source

As a simple, but important, illustration of the foregoing theory we consider the problem of an isotropic point source in a homogeneous medium. We first characterize the source by prescribing the initial values of the phase s and the amplitude coefficient z. The condition of isotropy implies that the limit $Z(0)$ of (7.3) is the same in all directions, i.e., on all rays. Then, if we denote distance from the source point P by r, (7.4) yields

$$z(r) = \frac{Z(0)}{r} . \tag{1}$$

From (6.2) we have

$$s(r) = s(P) + nr, \tag{2}$$

and from (5.1)

$$u = \frac{e^{ik[s(P)+nr]}}{r} Z(0) . \tag{3}$$

If the source is characterized by the inhomogeneous equation

$$\nabla^2 u + k^2 n^2 u = -\delta(X-P); \quad (n = \text{const.}) ; \tag{4}$$

and the radiation condition, it is well known that the unique solution is the <u>free space Green's function</u>,

$$u = \frac{e^{iknr}}{4\pi r} . \tag{5}$$

Comparing (3) and (5) we see that for a source characterized by (4) we should set

$$s(P) = 0, \; z_0(0) = \frac{1}{4\pi} . \tag{6}$$

The problem (4) is trivial if n is constant, for the exact solution is given by (5) and the asymptotic solution is unnecessary. However, if $n(X)$ is a variable index of refraction we may consider the non-trivial source problem

$$\nabla^2 u + k^2 n^2(X) u = -\delta(X-P). \tag{7}$$

If we assume that $s(P)$ and $z(0)$ are determined only by local properties, then these numbers are given by (6) and the asymptotic solution is given by (5.1), with the phase given on each ray emanating from P by (6.2):

$$s(\sigma) = s[x(\sigma)] = \int_0^{\sigma} n[X(\sigma')]d\sigma', \tag{8}$$

and the amplitude coefficient z given by (7.2):

$$z(\sigma) = \frac{1}{4\pi}\left[\frac{d\Omega}{da(\sigma)} \; \frac{n(P)}{n(\sigma)}\right]^{\frac{1}{2}} . \tag{9}$$

The problem we have solved here illustrates a general feature of our asymptotic method. The solution of the problem (7) was determined by our earlier considerations except for the values of $s(P)$ and $z(0)$ on each ray emanating from P. These values were determined from the exact solution of the simpler <u>canonical problem</u> (4). We shall frequently make use of a canonical problem, which can be solved exactly, to obtain certain undetermined coefficients for a more difficult problem which has the same local properties.

A9. Isotropic line source

In this section we examine the problem of an isotropic source, uniformly distributed on an infinite straight line in a homogeneous medium. Let r be the cylindrical co-ordinate measuring distance from the line. Since s_o is assumed to be constant on the source line, (6.4) implies that $\beta = \pi/2$, i.e., at every point on the source line, rays emanate at right angles to the line. From (7.17) we see that

$$\frac{1}{\rho_1} = -\frac{\dot{\beta}\sin\beta + \rho^{-1}\cos\beta}{\sin^2\beta} = 0, \tag{1}$$

for $\dot{\beta} = 0$ and the curvature ρ^{-1} of the source line is zero. Since the medium is homogeneous and the source is assumed isotropic and uniformly distributed, the resulting wave must be a function of r alone, hence (7.9) becomes

$$z(r) = r^{-\frac{1}{2}} \; z(0) \tag{2}$$

and (6.5) yields

$$s(r) = s_o + nr. \tag{3}$$

It follows from (5.1) that the wave produced by the given source is

$$u \sim \frac{z(0)}{\sqrt{r}}\, e^{ik(s_o+nr)} \; . \tag{4}$$

Let us now compare (4) with the two-dimensional free space Green's function $\frac{i}{4} H_o^{(1)}$ (knr) which is the solution of (8.5) in two dimensions. By employing the asymptotic expansion of the Hankel function we find that for knr $\to \infty$

$$\frac{i}{4} H_o^{(1)} (knr) \sim \frac{1}{2\sqrt{2\pi knr}}\, e^{i\frac{\pi}{4}}\, e^{iknr} \tag{5}$$

and we see that (4) and (5) agree exactly if we take

$$z(0) = \frac{1}{2\sqrt{2\pi kn}}\, e^{i\frac{\pi}{4}} \; , \quad s_o = 0. \tag{6}$$

We may now consider the problem of an isotropic line source in an inhomogeneous medium, characterized by the two-dimensional analogue of (8.8). In this case $X = x_1, x_2$, $n(X) = n(x_1, x_2)$, $P = p_1, p_2$ and δ is the two-dimensional delta function. The leading term of the asymptotic solution is obtained by setting

$$s_0 = 0 \quad \text{and} \quad Z_0(0) = a = \frac{1}{2\sqrt{2\pi k n(0)}}\, e^{i\frac{\pi}{4}} .$$

Then from (6.5) we see that on every ray $X = X(\sigma;\theta)$ emanating (at right angles) from the source line

$$s[X(\sigma;\theta)] = \int_0^\sigma n[X(\sigma';\theta)]\,d\sigma', \tag{7}$$

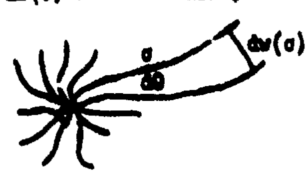
and from (7.6) we obtain

$$z_0[X(\sigma;\theta)] = a\left[\frac{d\eta\, d\theta\, n(0)}{da(\sigma)\,n(\sigma)}\right]^{1/2}. \tag{8}$$

Since n is independent of x_3, all rays remain in planes $x_3 = \text{const}$. Hence

$$da(\sigma) = dw(\sigma)\,dx_3 = dw(\sigma)\,d\eta. \tag{9}$$

The meaning of $dw(\sigma)$ is most easily seen from the following figure:



By inserting (9) in (8) and collecting our results we obtain

$$u \sim \frac{1}{2\sqrt{2\pi k}} \exp\left\{ik\int_0^\sigma n[X(\sigma',\theta)]\,d\sigma' + i\frac{\pi}{4}\right\}\left[\frac{d\theta}{n(\sigma)\,dw(\sigma)}\right]^{\frac{1}{2}} . \tag{10}$$

A 10. Reflection from a boundary

Let us suppose that a wave of the form (5.1) is incident on a boundary surface B; i.e., the rays associated with the wave intersect B. Let the solution u be required to satisfy the impedance boundary condition

$$\frac{\partial u}{\partial \nu} + ikz(X)\, u = 0, \quad X \text{ on } B. \tag{1}$$

Here $\frac{\partial u}{\partial \nu} = N \cdot \nabla u$ denotes the derivative of u along the outward normal N to B, and z is a given function called the impedance of the boundary. For $z = \infty$ and $z = 0$ (1) reduces to the simpler boundary conditions $u = 0$ and $\frac{\partial u}{\partial \nu} = 0$ respectively. We assume that a reflected wave, also of the form (5.1) is produced. To verify this assumption we shall show that (1) can be satisfied by the sum of the incident wave u^i and an appropriate reflected wave u^r which we shall construct. Thus we write u as

$$u = u^i + u^r \sim e^{iks^i} \sum_{m=0}^{\infty} z_m^i (ik)^{-m} + e^{iks^r} \sum_{m=0}^{\infty} z_m^r (ik)^{-m}. \tag{2}$$

We now insert (2) into (1) and obtain

$$e^{iks^i} \sum_m \left[\left(\frac{\partial s^i}{\partial \nu} + z\right) z_m^i + \frac{\partial z_{m-1}^i}{\partial \nu}\right](ik)^{-m+1} + e^{iks^r} \sum_m \left[\left(\frac{\partial s^r}{\partial \nu} + z\right) z_m^r + \frac{\partial z_{m-1}^r}{\partial \nu}\right](ik)^{-m+1} \sim 0; \quad X \text{ on } B. \tag{3}$$

In order that the two sums cancel each other, the exponentials must be equal, so we have

$$s^r(X) = s^i(X), \quad X \text{ on } B. \tag{4}$$

Then upon equating to zero the coefficients of each power of k in (3) we obtain

$$z_0^i\left(\frac{\partial s^i}{\partial \nu}+s\right)+z_0^r\left(\frac{\partial s^r}{\partial \nu}+s\right)=0;\ X \text{ on } B; \qquad (5)$$

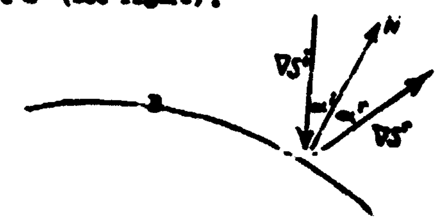
$$z_m^i\left(\frac{\partial s^i}{\partial \nu}+s\right)+z_m^r\left(\frac{\partial s^r}{\partial \nu}+s\right)+\frac{\partial z^i_{m-1}}{\partial \nu}+\frac{\partial z^r_{m-1}}{\partial \nu}=0;\ m \geq 1;\ X \text{ on } B. \qquad (6)$$

Equation (4) provides the value of s^r on B, (5) can be solved for z_0^r on B, and then (6) determines successively the z_m^r on B. It is clear that these values suffice to determine the reflected wave u^r. Let us now examine the properties of this wave.

Let $X = X_0(\eta_1, \eta_2)$ be the parametric equation of the boundary surface B. The results of section 6 could be used in conjunction with (4) to determine the reflected phase function $s^{(r)}(X)$, but it is more convenient to proceed independently: Differentiation of (4) with respect to η_1 and η_2 yields

$$\nabla s^r\cdot\frac{\partial X_0}{\partial \eta_j}=\nabla s^i\cdot\frac{\partial X_0}{\partial \eta_j};\ j=1,2;\ X \text{ on } B. \qquad (7)$$

Since $\frac{\partial X_0}{\partial \eta_1}$ and $\frac{\partial X_0}{\partial \eta_2}$ lie in the tangent plane to B at the point X, (7) shows that ∇s^i and ∇s^r have the same projection on the tangent plane. Therefore the plane containing this common projection and the normal N to B at X contains both ∇s^i and ∇s^r. In addition, since both s^i and s^r satisfy the eiconal equation (1.14), ∇s^i and ∇s^r have the same length, n(X). Since their tangential components are the same, their normal components must be of equal length. If they had the same sign, ∇s^i and ∇s^r would be identical, and this would violate the condition that s^r be outgoing from B. Therefore the normal components are of opposite sign. These results may be summarized in the <u>law of Reflection</u>. This states that the reflected ray direction ∇s^r lies in the plane containing the incident ray direction ∇s^i and the normal N to B, and that the angle of reflection α^r equals the angle of incidence α^i (See figure).



The equality of α_i and α_r follows from the fact that the normal components of ∇s^i and ∇s^r have the same length.

The initial direction of each reflected ray is determined by the law of reflection, and the initial value of s^r is given by (4). We now set $\alpha = \alpha^i = \alpha^r$, and note that $\frac{\partial s^r}{\partial \nu} = n \cos \alpha$ and $\frac{\partial s^i}{\partial \nu} = -n \cos \alpha$. Then (5) and (6) yield, for the initial values of s_m^r,*

$$s_0^r = \frac{n \cos\alpha - 1}{n \cos\alpha + 1} \; s_0^i \; ; \; X \text{ on } B; \tag{8}$$

$$s_m^r = \frac{n \cos\alpha - 1}{n \cos\alpha + 1} \; s_m^i - \frac{1}{n \cos\alpha + 1} \left(\frac{\partial s^i_{m-1}}{\partial \nu} + \frac{\partial s^r_{m-1}}{\partial \nu} \right); \; m \geq 1; \; X \text{ on } B. \tag{9}$$

These initial values enable us to construct the reflected wave, thus verifying our assumption that a solution of the form (2) satisfies (1).

We have seen that if the region of space under consideration in a problem has a boundary it acts as a secondary surface source and produces a reflected wave. We have just computed the initial conditions which determine this wave. The reflected wave may be reflected again from another part of the boundary and this may occur any number of times. All these singly and multiply reflected waves must be included in the sum of waves forming the asymptotic solution of the problem. Some problems involve, instead of a boundary, an <u>interface</u>, which is a surface S across which n(X) may be discontinuous, and on which the solution u(X)

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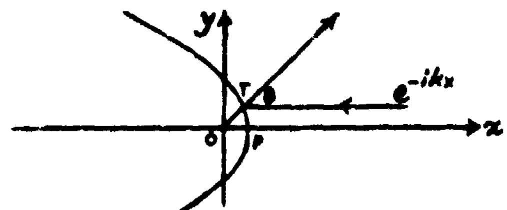
* The factor $\frac{n \cos \alpha - 1}{n \cos \alpha + 1}$ in (8) may be called a <u>reflection coefficient</u>.

must satisfy appropriate continuity conditions. In such cases if a wave is incident upon S that surface acts as a secondary source and produces not only a reflected wave but also a <u>transmitted</u> wave on the other side of the interface. In addition if the boundary or interface contains edges or vertices, they will act as secondary line and point sources, respectively, producing what we shall call <u>diffracted waves</u>. All singly and multiply reflected, transmitted, and diffracted waves must be included in the sum of waves forming the asymptotic solution. In subsequent sections we shall show how to calculate these waves.

A 11. <u>Reflection by a parabolic cylinder</u>

Before considering transmitted and diffracted waves, we shall illustrate the results of the preceding section by considering the problem of reflection of a plane wave e^{-ikx}, incident along the axis of a parabolic cylinder, from the outside.



We will take the index of refraction to be $n=1$ and the boundary condition to be $u = 0$. The incident rays are parallel to the axis, and by the well-known focussing property of parabolas, the reflected rays are radial lines which would pass through the focus if extended backward. Therefore the reflected wave fronts are the circular cylinders $r = \sqrt{x^2 + y^2} = \text{const.}$, i.e. the reflected wave is a <u>cylindrical wave</u>.

In general, for cylindrical waves, one principal radius of curvature of the wave front $r = \text{const.}$ is infinite and the other equals r. If we take the wave-front $w(\sigma_1)$ of section 4 to be $r = 0$, then $\sigma = r$, $\rho_1 = 0$, $\rho_2 = \infty$, and (4.5,6) become

$$z_0(r,\theta) = u_0(r_0,\theta)\left(\frac{r_0}{r}\right)^{\frac{1}{2}} \tag{1}$$

$$z_m(r,\theta) = z_m(r_0,\theta)\left(\frac{r_0}{r}\right)^{\frac{1}{2}} - \frac{1}{2r^{1/2}} \int_{r_0}^{r} (r')^{1/2} \Delta z_{m-1}(r',\theta)\,dr'; \quad m=1,2,\dots \tag{2}$$

By using (1) and (2) we find by induction that

$$z_m(r,\theta) = \sum_{j=0}^{m} f_{jm}(\theta)\, r^{-(\frac{1}{2})-j} \quad . \tag{3}$$

Inserting (3) into (2) yields the recursive formulas for $f_{jm}(\theta)$,

$$f_{jm}(\theta) = \frac{1}{2j}\left[(j-\tfrac{1}{2})^2 f_{j-1,m-1} + f''_{j-1,m-1}\right], \quad j \neq 0,\ m \geq 1; \tag{4}$$

$$f_{0m}(\theta) = r_0^{1/2}(\theta) z_m\left[r_0(\theta),\theta\right] - \sum_{j=1}^{m} r_0^{-j} f_{jm}, \quad m \geq 1; \tag{5}$$

$$f_{00}(\theta) = z_0\left[r_0(\theta),\theta\right]\left[r_0(\theta)\right]^{1/2}. \tag{6}$$

$z_m\left[r_0(\theta),\theta\right]$ is the value of z_m at some point $r_0(\theta)$ on the ray θ = const. For a cylindrical wave $s = r + s_0$ (s_0 = const.). Thus (5,1) and (3) yield

$$u \sim \frac{e^{ik(r+s_0)}}{\sqrt{r}} \sum_{m=0}^{\infty} (ik)^{-m} \sum_{j=0}^{m} f_{jm}(\theta) r^{-j} \ . \tag{7}$$

Returning now to our reflection problem, we write the equation of the parabola of focal length p as

$$r = r_o(\theta) = \frac{2p}{1+\cos\theta} = p \sec^2 \frac{\theta}{2} \, . \tag{8}$$

On the parabola, the incident field is $e^{-ikx} = e^{-ikr_o \cos\theta}$. Therefore by inserting the total field, incident plus reflected, into the boundary condition $u = 0$, we obtain

$$e^{-ikr_o \cos\theta} + e^{iks(r_o,\theta)} \sum_{m=0}^{\infty} z_m\left[r_o(\theta), \theta\right](ik)^{-m} \sim 0. \tag{9}$$

If we equate to zero coefficients of powers of k in (9) we find that

$$s(r_o,\theta) = -r_o(\theta) \cos\theta, \tag{10}$$

$$z_o\left[r_o(\theta), \theta\right] = -1 \tag{11}$$

$$z_m\left[r_o(\theta), \theta\right] = 0; \; m \geq 1. \tag{12}$$

From (10) and (8) we see that on each ray

$$s = s(r_o,\theta) + r - r_o = r - r_o(1+\cos\theta) = r - 2p, \tag{13}$$

hence in (7),

$$s_o = -2p. \tag{14}$$

By using (11) in (6) we obtain

$$f_{oo}(\theta) = -\left[r_o(\theta)\right]^{\frac{1}{2}} = -p^{\frac{1}{2}} \sec \frac{\theta}{2}, \tag{15}$$

hence from (3),

$$z_o(r,\theta) = -p^{1/2}(\sec\frac{\theta}{2})r^{-\frac{1}{2}}. \tag{16}$$

Now we may determine the $f_{jm}(\theta)$ from (4) and (5), using (12) and (15). By calculating the first few f_{jm} we find that they have the form

$$f_{jm}(\theta) = a_{jm}p^{\frac{1}{2}+j-m}(\sec\frac{\theta}{2})^{2j+1}, \tag{17}$$

and (17) can be proved by induction to hold generally. In (17) the a_{jm} are constants which satisfy the recursion formulas

$$a_{jm} = \frac{4}{2}(j-\frac{1}{2})\, a_{j-1,m-1};\ j\geq 1,\ m\geq 1; \tag{18}$$

$$a_{om} = -\sum_{j=1}^{m} a_{jm},\quad m\geq 1 \tag{19}$$

$$a_{oo} = -1. \tag{20}$$

From (18-20) the a_{jm} can be determined successively.

Collecting our results we have, for the asymptotic expansion of the reflected wave,

$$u \sim e^{ik(r-2p)} \sum_{m=0}^{\infty} (ikp)^{-m} \sum_{j=0}^{m} a_{jm}(pr^{-1}\sec^2\frac{\theta}{2})^{j+\frac{1}{2}}. \tag{21}$$

The problem treated asymptotically in this section can be solved exactly by separation of variables in parabolic co-ordinates. When the exact solution is expanded asymptotically for large k it yields precisely (21). If the exact solution is compared with the first few terms of (21) (through $(kp)^{-2}$), the numerical agreement is found to be good for

$$kp \geq 2. \tag{22}$$

The problem we have discussed here, and numerous related problems are treated in [18]. The domain of validity (22) of the leading terms of the expansion is typical of most problems where comparisons with exact solutions have been made.

A 12. <u>Reflection and transmission at an interface</u>

In this section we shall determine the secondary waves which are produced when a wave u^i is incident on one side of a surface S, across which the index of refraction, $n(X)$ may have a jump discontinuity. Such a surface is called an <u>interface</u>. We denote by $n_1(X)$ and $n_2(X)$ the limiting values of n as S is approached from sides 1 and 2 respectively, and we require the solution u to satisfy the two boundary conditions

$$u^1 = au^2, \quad \frac{\partial u^1}{\partial \nu} = b\frac{\partial u^2}{\partial \nu}, \quad \text{on } S. \tag{1}$$

Here a and b are given functions on S and $\frac{\partial u}{\partial \nu} = N \cdot \nabla u$ denotes the normal derivative to S.

We assume that on the side of S, say side 1, from which the wave u^i is incident, a reflected wave u^r is produced; and on the other side a transmitted wave, u^t is produced. To verify this assumption we shall show that (1) is satisfied by

$$u^1 = u^i + u^r, \quad u^2 = u^t, \tag{2}$$

where u^r and u^t are appropriately constructed waves, outgoing from S. Thus we set

$$u^i \sim e^{iks^i} \sum z^i_m (ik)^{-m}, \quad u^r \sim e^{iks^r} \sum z^r_m (ik)^{-m}, \quad u^t \sim e^{iks^t} \sum z^t_m (ik)^{-m}, \tag{3}$$

where $z^i_m = z^r_m = z^t_m = 0$ for $m = -1, -2, \ldots$. We insert (3) into (2) and (2) into (1), and derive the following conditions.

$$s^i(X) = s^r(X) = s^t(X); \quad X \text{ on } S; \tag{4}$$

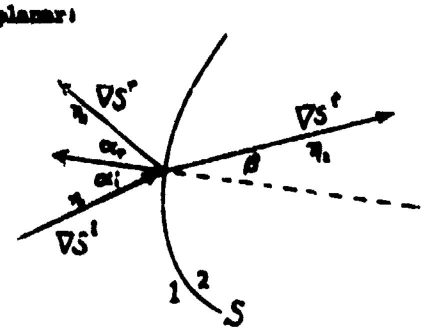
$$z^i_m + z^r_m = a z^t_m \text{ ; } X \text{ on } S; \tag{5}$$

$$\frac{\partial s^i}{\partial \nu} z^i_{m+1} + \frac{\partial z^i_m}{\partial \nu} + \frac{\partial s^r}{\partial \nu} z^r_{m+1} + \frac{\partial z^r_m}{\partial \nu} = b\left[\frac{\partial s^t}{\partial \nu} z^t_{m+1} + \frac{\partial z^t_m}{\partial \nu}\right]; \; X \text{ on } S. \tag{6}$$

If $X = X_o(\eta_1, \eta_2)$ is the parametric equation of the interface, S, then differentiation of (4) yields

$$\nabla s^i \cdot \frac{\partial X_o}{\partial \eta_j} = \nabla s^r \cdot \frac{\partial X^o}{\partial \eta_j} = \nabla s^t \cdot \frac{\partial X^o}{\partial \eta_j} \; ; \; j = 1,2; \; X \text{ on } S, \tag{7}$$

and these equations imply that ∇s^i, ∇s^r, and ∇s^t have the same projection on the tangent plane to S at the point X. It follows that these three vectors and the unit normal vector N are coplanar:



From the eiconal equation, we note that

$$(\nabla s^i)^2 = n_1^2\,,\ (\nabla s^r)^2 = n_1^2\,,\ (\nabla s^t)^2 = n_2^2\,. \tag{8}$$

It follows now from (7),(8), and the outgoing condition, that ∇s^r and ∇s^t are directed as shown in the figure and that

$$\alpha^r = \alpha^i;\ n_2 \sin\beta = n_1 \sin\alpha_i. \tag{9}$$

We set

$$\alpha^i = \alpha^r = \alpha, \tag{10}$$

and thus

$$\sin\beta = \frac{n_1}{n_2}\sin\alpha. \tag{11}$$

Equation (10) is the familiar law of reflection, and (11) is the <u>law of refraction</u>.

Returning to (6) we note that

$$\frac{\partial s^r}{\partial \nu} = n_1 \cos\alpha,\ \frac{\partial s^i}{\partial \nu} = -n_1\cos\alpha,\ \frac{\partial s^t}{\partial \nu} = -n_2 \cos\beta. \tag{12}$$

We insert (12) into (6) and introduce the ratio

$$z = \frac{bn_2 \cos\beta}{an_1 \cos\alpha}\,. \tag{13}$$

Then (5) and (6) take the form

$$a z_m^t - z_m^r = z_m^i, \quad X \text{ on } S, \tag{14}$$

$$a z_m^t + z_m^r = z_m^i + \frac{1}{n_1 \cos\alpha}\left[b\,\frac{\partial z_{m-1}^t}{\partial\nu} - \frac{\partial z_{m-1}^i}{\partial\nu} - \frac{\partial z_{m-1}^r}{\partial\nu}\right], \quad X \text{ on } S, \tag{15}$$

and these equations are easily solved to yield

$$z_m^r = \frac{1-z}{1+z}\, z_m^i + \frac{1}{(1+z)n_1 \cos\alpha}\left[b\,\frac{\partial z_{m-1}^t}{\partial\nu} - \frac{\partial z_{m-1}^i}{\partial\nu} - \frac{\partial z_{m-1}^r}{\partial\nu}\right], \quad X \text{ on } S, \tag{16}$$

$$z_m^t = \frac{2}{a(1+z)}\, z_m^i + \frac{1}{a(1+z)n_1 \cos\alpha}\left[b\,\frac{\partial z_{m-1}^t}{\partial\nu} - \frac{\partial z_{m-1}^i}{\partial\nu} - \frac{\partial z_{m-1}^r}{\partial\nu}\right], \quad X \text{ on } S. \tag{17}$$

(16) and (17) are valid for $m = 0,1,2, \ldots$, but in each case the second term on the right side vanishes for $m = 0$.*

The initial values of s^r and s^t on each reflected and transmitted ray are given by (4), and the initial values of the coefficients z_m^r and z_m^t are given by (16) and (17). These initial values enable us to construct the reflected and transmitted waves, thus verifying our initial assumption.

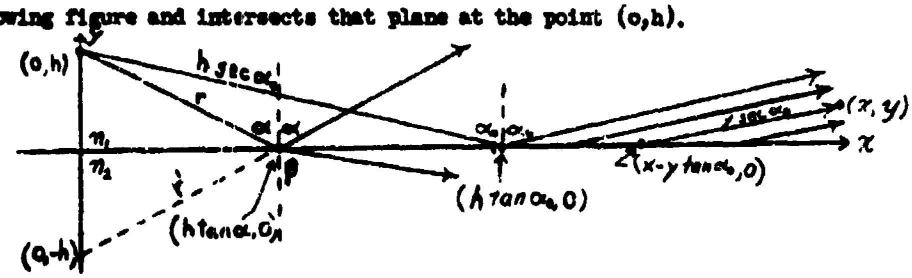
In the foregoing discussion we have tacitly assumed that (9) can be solved to obtain the angle of refraction, β. For all angles of incidence $\alpha < \pi/2$, this is certainly true provided $n_1/n_2 \leq 1$. However, for $n_1/n_2 > 1$ there is a critical angle of incidence α_0 for which $\frac{n_1}{n_2} \sin\alpha_0 = 1$. For this angle of incidence, the angle of refraction is $\beta_0 = \pi/2$ and the corresponding transmitted ray is tangent to the interface. Furthermore for $\alpha > \alpha_0$ (11) has no real solution β and our earlier discussion must be modified.

These complications, which occur when $n_1/n_2 > 1$ are associated with the phenomenon of total reflection. They will be discussed further, in a special case, in the following section.

*For $m = 0$, the factors $\frac{1-z}{1+z}$ and $\frac{2}{a(1+z)}$ may be called reflection and transmission coefficients.

A 13. Reflection and transmission of a cylindrical wave at a plane interface

Let $n(x,y,z) = n_1$ for $y > 0$ and $n(x,y,z) = n_2$ for $y<0$, where n_1 and n_2 are given constants. On the interface, $y = 0$ we prescribe the boundary conditions (12.1) with a and b constant. The incident wave is assumed to be excited by a line source. The line is perpendicular to the plane of the following figure and intersects that plane at the point (o,h).



Let $r = \sqrt{x^2+(y-h)^2}$ denote distance from the source. Then, if we assume that the source produces the incidence field $u^i = \frac{i}{4} H_0^{(1)}(kn_1 r)$, (See section A.9) the incident wave is given by*

$$u^i \sim e^{ikn_1 r} \sum_{m=0}^{\infty} s_m^i(r)(ik)^{-m}; \; y \geq 0; \tag{1}$$

where

$$s_0^i(r) = \frac{a_1}{\sqrt{2n_1 r}} \; ; \; s_m^i(r) = \frac{a_1 \prod_{j=1}^{m} (j-\frac{1}{2})^2}{m!\,(2n_1 r)^{m+\frac{1}{2}}} ; m=1,2,\ldots ; \; a_1 = \frac{e^{i\pi/4}}{2\sqrt{\pi k}} . \tag{2}$$

Because of the symmetry with respect to the y-axis we need calculate all functions only for $x \geq 0$.

From the law of reflection it is easily seen that the phase s^r of the reflected wave is $s^r = n_1 r'$ where $r' = \sqrt{x^2+(y+h)^2}$ is the distance from the point (0,-h) representing the "image" of the source. Therefore the reflected

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*(1) and (2) are obtained from the asymptotic expansion of the Henkel function $H_n^{(1)}(r)$ for $x \to +\infty$.

wave is given by

$$u^r \sim e^{ikn_1 r'} \sum_{m=0}^{\infty} z_m^r (ik)^{-m}; \quad y \geq 0. \tag{3}$$

Since the reflected wave-fronts are circular cylinders, the z_m^r may be obtained from (11.3) - (11.6) by replacing r by r' and k by kn_1. We also set $r_o = h \sec \alpha$ because, for points on the interface, $r' = h \sec \alpha$. Thus

$$z_m^r(r', \alpha) = (n_1)^{-m} \sum_{j=0}^{m} f_{jm}(\alpha)(r')^{-j-\frac{1}{2}}; \tag{4}$$

$$f_{jm}(\alpha) = \frac{1}{2j}\left[(j-\tfrac{1}{2})^2 f_{j-1,m-1} + f''_{j-1,m-1}\right], \; j \neq 0, \; m \geq 1; \tag{5}$$

$$f_{om}(\alpha) = (h \sec \alpha)^{\frac{1}{2}} z_m^r (h \sec \alpha, \alpha) - \sum_{j=1}^{m} (h \sec \alpha)^{-j} f_{jm}(\alpha), \; m \geq 1; \tag{6}$$

$$f_{oo}(\alpha) = z_o^r(h \sec \alpha, \alpha)\,(h \sec \alpha)^{1/2}. \tag{7}$$

The phase of the incident wave at the point $(h \tan \alpha, o)$ is $s^i = n_1 h \sec \alpha$. Therefore the phase of the transmitted wave at a distance σ from that point (along the transmitted ray) is $s^t = n_1 h \sec \alpha + n_2 \sigma$. Thus the transmitted wave is given by

$$u^t \sim e^{ik(n_1 h \sec \alpha + n_2 \sigma)} \sum_{m=0}^{\infty} z_m^t (ik)^{-m}. \tag{8}$$

The parametric equation for the transmitted ray is

$$X = (x,y) = (h \tan \alpha, o) + \sigma(\sin \beta, -\cos \beta) \tag{9}$$

and from the law of refraction, (12.11),

$$\sin \beta = \mu^{-1} \sin \alpha; \ \mu = \frac{n_2}{n_1} = \frac{c_1}{c_2}. \tag{10}$$

Since the media are homogeneous, we may use (4.6) to determine the functions z_m^t. Because the entire problem is independent of z, we may take one radius of curvature, say ρ_2, to be infinite. To determine ρ_1 we must find the caustic of the transmitted rays, i.e. the envelope of the family of straight lines (9). Using β as a parameter, we denote the caustic curve by $X = X(\beta)$. ρ_1 will be the distance from the point ($h \tan \alpha, o$) to the caustic, along the backward extension of the transmitted ray. Thus by setting $\sigma = -\rho_1$ in (9) we obtain

$$X(\beta) = (h \tan \alpha, 0) - \rho_1(\sin \beta, -\cos \beta), \tag{11}$$

and differentiation with respect to β yields

$$\frac{dX}{d\beta} = (h \sec^2 \alpha \ \frac{d\alpha}{d\beta}, 0) - \rho_1(\cos \beta, \sin \beta) - \frac{d\rho_1}{d\beta} (\sin \beta, -\cos \beta). \tag{12}$$

Since the vector $\frac{dX}{d\beta}$ is tangent to the caustic, it is parallel to the ray, hence perpendicular to the vector ($\cos \beta, \sin \beta$). It follows that

$$0 = \frac{dX}{d\beta} \cdot (\cos \beta, \sin \beta) = h \sec^2 \alpha \cos \beta \frac{d\alpha}{d\beta} - \rho_1 . \tag{13}$$

But from (10),

$$\cos \beta \frac{d\beta}{d\alpha} = \mu^{-1} \cos \alpha, \tag{14}$$

hence

$$\rho_1 = \mu h \sec^3 \alpha \cos^2 \beta = \mu^{-1} h \sec^3 \alpha(\mu^2 - \sin^2 \alpha). \tag{15}$$

This value of ρ_1 is to be used in the following equation, obtained from (4.6):

$$z_m^t(\sigma) = \left(1 + \frac{\sigma}{\rho_1}\right)^{-1/2}\left(z_m^t(o) - \frac{1}{2n_2}\int_0^\sigma \left(1 + \frac{\sigma'}{\rho_1}\right)^{1/2} \Delta z_{m-1}^t(\sigma')\, d\sigma'\right); \quad m=0,1,2,\ldots \tag{16}$$

As usual, we take $z_{-1}^t = 0$.

The functions z_m^t and z_m^r are given by (16) and (4.7) once the initial values $z_m^t(o)$ and z_m^r (h sec α,α) are specified. However, these values are given by (12.16) and (12.17). Thus

$$z_m^r(h\sec\alpha,\alpha) = \frac{1-z}{1+z}\, z_m^i(h\sec\alpha) + \frac{1}{(1+z)n_1\cos\alpha}\left[b\,\frac{\partial z_{m-1}^t}{\partial y} - \frac{\partial z_{m-1}^i}{\partial y} - \frac{\partial z_{m-1}^r}{\partial y}\right]_{\substack{x=h\tan\alpha\\ y=0}} \tag{17}$$

$$z_m^t(o) = \frac{2}{a(1+z)}\, z_m^i(h\sec\alpha) + \frac{1}{a(1+z)n_1\cos\alpha}\left[b\,\frac{\partial z_{m-1}^t}{\partial y} - \frac{\partial z_{m-1}^i}{\partial y} - \frac{\partial z_{m-1}^r}{\partial y}\right]_{\substack{x=h\tan\alpha\\ y=0}} . \tag{18}$$

Here

$$z = \frac{b n_2\cos\beta}{a n_1\cos\alpha} = \frac{b\sqrt{\mu^2-\sin^2\alpha}}{a\cos\alpha}. \tag{19}$$

From (2), (4), (7), and (17) we easily obtain

$$z_o^i = \frac{a_1}{(2n_1 r)^{1/2}}, \qquad r = \sqrt{x^2+(y-h)^2}\,; \tag{20}$$

$$z_o^r = \frac{a_1}{(2n_1)^{1/2}}\,\frac{1-z}{1+z}(r')^{-1/2}, \quad r' = \sqrt{x^2+(y+h)^2}, \quad \alpha = \text{arc tan}\,\frac{x}{y+h}\,; \tag{21}$$

and from (16) and (18),

$$z_o^t = \frac{2a_1}{a(1+z)}\left[2n_1 h\sec\alpha\left(1 + \frac{\sigma}{\rho_1}\right)\right]^{-\frac{1}{2}}. \tag{22}$$

(20) and (21) give s_0^i and s_0^r explicitly as functions of x and y. To obtain $s_0^t(x,y)$ from (22) it would be necessary to obtain $\sigma(x,y)$ and $\alpha(x,y)$ from (9) and (10).

In order to facilitate the computation of $\left.\frac{\partial s_0^t}{\partial y}\right|_{y=0}$, which will be needed shortly, it is convenient to simplify (22). We first note, from (9) and (10) that

$$\sigma = \frac{\mu}{\sin\alpha}(x-h\tan\alpha) = \frac{\mu x}{\sin\alpha} - \frac{\mu h}{\cos\alpha} \; . \qquad (23)$$

From (15) and (23) we now obtain

$$\sigma+\rho_1 = \frac{\mu x}{\sin\alpha} + \frac{\mu h}{\cos\alpha}\left[-1+\sec^2\alpha-\mu^{-2}\tan^2\alpha\right] = \frac{\mu}{\sin\alpha}\left[x+h(1-\mu^{-2})\tan^3\alpha\right]. \qquad (24)$$

We now insert (15) and (24) in (22). This yields

$$s_0^t = \frac{2m_1}{a(1+s)}\left[\frac{\rho_1}{2m_1 h\sec^3\alpha\,(\rho_1+\sigma)}\right]^{1/2} = \frac{2m_1}{a(1+s)}\left[\frac{\cos^2\beta\sin\alpha}{2m_1\cos^2\alpha\left[x+h(1-\mu^{-2})\tan^3\alpha\right]}\right]^{1/2}$$

$$= \frac{m_1}{a(1+s)}\,\frac{\cos\beta}{\cos\alpha}\left[\frac{2\sin\alpha}{m_1\left[x+h(1-\mu^{-2})\tan^3\alpha\right]}\right]^{1/2}. \qquad (25)$$

From (9) we note that

$$y = -\sigma\cos\beta, \qquad x = h\tan\alpha + \sigma\sin\beta, \qquad (26)$$

and differentiation with respect to y yields

$$1 = -\frac{\partial\sigma}{\partial y}\cos\beta - \sigma\frac{\partial\cos\beta}{\partial y}\, . \qquad (27)$$

Hence, when y = 0 (10), (26) and (27) imply

$$\beta = \pi/2,\ \sin\alpha = \mu,\ \alpha = \alpha_o = \sin^{-1}\mu,\ \tan^2\alpha = \frac{1}{\mu^{-2}-1},\ z = 0,\ \frac{\partial \cos\beta}{dy} = \frac{1}{\sigma} = -\frac{1}{x-h\tan\alpha_o}. \tag{28}$$

We may now use (28) and (25) to find $\frac{\partial z_o^t}{dy}\Big|_{y=0}$. The computation is greatly simplified by noting that $\cos\beta = 0$ when $y = 0$, hence

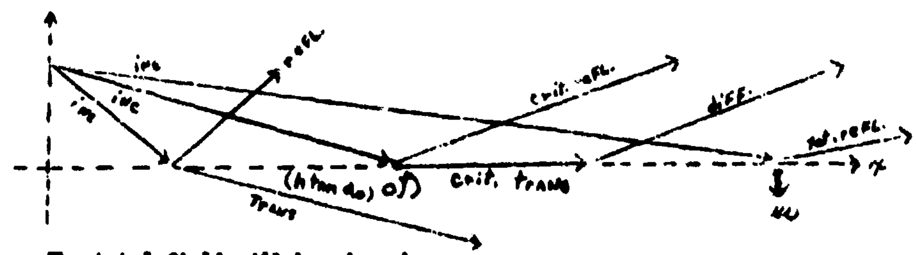
$$\frac{\partial z_o^t}{dy}\Big|_{y=0} = \frac{a_1}{\pi(1+\pi)\cos\alpha}\left[\frac{2\sin\alpha}{n_1[x+h(1-\mu^{-2})\tan^3\alpha]}\right]^{1/2}\frac{\partial\cos\beta}{dy}\Big|_{y=0}$$

$$= -\frac{a_1}{a\cos\alpha_o}\left(\frac{2\mu}{n_1}\right)^{1/2}(x-h\tan\alpha_o)^{-3/2}. \tag{29}$$

The derivatives $\frac{\partial z_o^i}{dy}$ and $\frac{\partial z_o^r}{dy}$ can be obtained directly from (20) and (21). Then $z_1^r(h\sec\alpha,\alpha)$ and $z_1^t(0)$ can be obtained from (17) and (18) and used in (4-7) and (16) to find z_1^r and z_1^t along their respective rays. We will not carry out this calculation here. The function z_1^r has been computed in [18].

If $\mu < 1$ total reflection occurs for rays incident at angles α greater than the critical angle $\alpha_o = \sin^{-1}\mu$. The transmitted rays corresponding to angles of incidence α in the interval $0 \leq \alpha \leq \alpha_o$ cover the entire lower half-space, and the critically transmitted ray, for which $\beta = \pi/2$, lies in the interface $y = 0$. For rays incident at angles $\alpha > \alpha_o$ the corresponding reflected rays are called "totally reflected rays", and the corresponding angle of refraction, β, is complex. Hence no real transmitted rays originate in the "region of total reflection", $x > h\tan\alpha_o$, of the interface. If any wave is produced in the space below this region it is present in the same region as the transmitted wave, whose rays originate in the "region of regular reflection", $0 \leq x \leq h\tan\alpha_o$. We will see that a wave is produced below the region of total reflection. This wave u^e is called the evanescent wave because its amplitude decays rapidly with distance from the interface. The wave u^e together with the

the incident wave u^i and the reflected wave u^r, satisfies the boundary conditions in the region of total reflection, but the additional presence of the transmitted wave u^t below this portion of the interface requires that a fifth wave, the diffracted wave u^d, be produced above the region of total reflection. Then u^d together with u^t satisfies the boundary conditions. Typical rays associated with the various waves are indicated in the following figure.



The total field will be given by

$$u \sim u^i + u^r + u^d \; ; \quad y > 0; \tag{30}$$

$$u \sim u^t + u^e \quad , \quad y < 0; \tag{31}$$

but u^d and u^e are zero in the region of regular reflection.

We have already seen how to obtain u^t and the regularly reflected part of u_r. In the same way u^e and the totally reflected part of u^r satisfy (17) and (18) (with t replaced by e), but the angle of refraction β is complex; therefore the determination of u^e below the interface is different from the determination of u^t. Since u^e is not required for the calculation of s_o^r, that function can be found as before, i.e., (21) remains valid in the region of total reflection. For $m>0$ u^e is required to determine s_m^r. However, we will not determine u^e here. We only remark that it decays exponentially with distance from the interface.

The wave u^d is completely determined by u^t and the boundary conditions. To calculate u^d we set

$$u^d \sim e^{iks^d} \sum_{m=0}^{\infty} z_m^d (ik)^{-m}, \quad u^t \sim e^{iks^t} \sum_{m=0}^{\infty} z_m^t (ik)^{-m}, \tag{32}$$

and write the boundary conditions (12.1) in the form

$$u^d = au^t, \quad \frac{\partial u^d}{\partial y} = b\,\frac{\partial u^t}{\partial y}, \quad y=0,\ x > h \tan \alpha_0. \tag{33}$$

By inserting (32) in (33), in the usual way, we obtain

$$s^d(x,0) = s^t(x,0) = n_1 h \sec \alpha_0 + n_2(x-h \tan \alpha_0), \quad x > h \tan \alpha_0; \tag{34}$$

$$z_m^d(x,0) = az_m^t(x,0) \qquad , \ x > h \tan \alpha_0; \tag{35}$$

$$\frac{\partial s^d}{\partial y} z_{m+1}^d + \frac{\partial z_m^d}{\partial y} = b\,\frac{\partial z_m^t}{\partial y} \qquad y=0,\ x > h \tan \alpha_0. \tag{36}$$

In (36) we have used the fact that $\frac{\partial s^t}{\partial y} = \nabla s^t \cdot (0,1) = 0$ when $y=0$, $x > h \tan \alpha_0$. This follows immediately from the direction of the critically transmitted ray.

To calculate u^d we must first solve the initial value problem for $s^d(x,y)$ in $y > 0$ using the initial values (34) on the surface $y=0$. Proceeding as in section 6, we easily find that all of the diffracted rays are parallel to the critically reflected ray and leave the interface at the angle α_0 to the normal. We also find that

$$s^d = n_1(h+y) \sec \alpha_0 + n_2[x-(h+y) \tan \alpha_0]; \ y \geq 0 \quad x \geq (h+y)\tan \alpha_0. \tag{37}$$

Since the wave fronts of u^d are planes, u^d is called a general plane wave. The adjective "general" is required because the amplitude is not constant. To determine the amplitude coefficients z_m^d we note first from (25) that $z_o^t(x,0) = 0$ because the factor $\cos\beta$ vanishes when $\beta = \pi/2$. It then follows from (35) and (4.6) that $z_o^d(x,0) = 0$, hence

$$z_o^d(x,y) = 0. \tag{38}$$

From (4.6) we see that z_1^d is constant on each diffracted ray, because the radii of curvature ρ_1 and ρ_2 are both infinite for general plane waves. Hence from (35)

$$z_1^d(x,y) = z_1^d(x-y\tan\alpha_o,0) = az_1^t(x-y\tan\alpha_o,\ 0),\ y \geq 0,\ x \geq (h+y)\tan\alpha_o. \tag{39}$$

Since we have not computed z_1^t we cannot use (39) to obtain z_1^d. However, if we set m=0 in (36) and note that $\frac{\partial s^d}{\partial y} = \nabla s^d \cdot (0,1) = n_1\cos\alpha_o$, we obtain

$$z_1^d(x,0) = \frac{b}{n_1\cos\alpha_o}\,\frac{\partial z_o^t(x,0)}{\partial y}\ , \tag{40}$$

and (39) and (40) yield

$$z_1^d(x,y) = \frac{b}{n_1\cos\alpha_o}\,\frac{\partial z_o^t(x',y')}{\partial y}\Bigg|_{\substack{y'=0\\ x'=x-y\tan\alpha_o}}. \tag{41}$$

Finally, we insert (29) in (41). The result is

$$z_1^d(x,y) = \frac{a_2 b\sqrt{\pi\mu}}{a(1-\mu^2)}\left\{n_1\left[x-(h+y)\tan\alpha_o\right]\right\}^{-3/2},\ y \geq 0,\ x \geq (h+y)\tan\alpha_o\ . \tag{42}$$

The term $e^{iks^d}(ik)^{-1}z_1^d$ is the leading term of u^d. It is of order $1/k$ with respect to the incident field, and of the same order as the term u_1^r, which we have not determined. z_1^d becomes infinite on the diffracted ray which coincides with the critically reflected ray,

$$x = (h+y)\tan \alpha_o. \tag{43}$$

Therefore the present asymptotic expansion fails on that ray.

We note that the diffracted wave is completely determined by its initial values, all of which are given by (35). Hence all the quantities in (36) are already determined and therefore that equation must be an identity.

A 14 Diffraction by Edges and Vertices

A surface or curve is regular at a point if it can be represented by functions which have derivatives of all orders in a neighborhood of the point. An edge is a curve, on a boundary or interface, which forms a locus of points where the surface is not regular. A vertex is an isolated point, on a boundary or interface, where the surface is not regular, or an isolated point on an edge where the edge is not regular. Examples of edges and vertices are: the edges and vertices of a polyhedral interface or boundary, the vertex of a conical interface or boundary, the edges of an aperture in a thin screen. (In the last case the screen is a boundary surface, but both sides of the screen being connected by the aperture, form the domain of the problem. If the aperture edge is not regular it contains vertex points.)

We have already mentioned diffracted waves and diffracted rays. We will use these terms to include all waves and rays not predicted by the classical theory of geometrical optics. When any wave, u^i is incident upon an edge or vertex, M, we assume that M acts as a secondary source manifold producing a diffracted wave

$$u^d \sim e^{iks^d} \sum_{m=0}^{\infty} z_m^d (ik)^{-m}. \tag{1}$$

By analogy with our results for secondary waves produced by reflection and transmission, we assume that

$$s^d(X) = s^i(X) , \quad X \text{ on } M, \tag{2}$$

where s^i is the phase of u^i. The point or curve, M, is a caustic of the diffracted wave; hence, as we have seen, the functions z_m^d are infinite there. However, the limit $\hat{z}^d$, introduced in section A7, is finite. We assume that $\hat{z}^d$ is proportional to the amplitude z^i of the incident wave at X, i.e.,

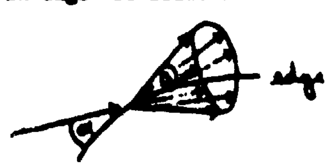
$$\hat{z}^d(X) = (d)\, z^i(X); \quad X \text{ on } M. \qquad (3)$$

The proportionality factor (d) will be called a <u>diffraction coefficient</u>. It is analogous to the <u>reflection coefficient</u> $r = \frac{n \cos \alpha - s}{n \cos \alpha + s}$, which appears in (10.8), or the <u>transmission coefficient</u> $t = \frac{2}{s(1+s)}$, which appears in (12.17) for $m = 0$. In general, diffraction coefficients, unlike reflection and transmission coefficients, cannot be obtained directly from the prescribed boundary conditions. Instead they are obtained either from the solution of canonical problems or by boundary layer methods [3]. The latter methods also yield the values of the z_m^d for $m > 0$. For some purposes, it might suffice to determine (d) experimentally, but this has not yet been attempted. In general, the diffraction coefficient depends on the local geometric properties of M, the local values of the index of refraction, the directions of both incident and diffracted rays, and the wave number, k; and it vanishes in the limit $k \to \infty$.

The phase $s^d(X)$ and the rays of the diffracted wave are obtained by solving the initial value problem for the eiconal equation, with initial values given by (2). Since this has been discussed in detail in Section A6 we need only mention the consequences of the special form (2) of the initial values when M is an edge. Let us first assume that the index of refraction is continuous in a neighborhood of the edge, as is the case when the edge lies on a boundary surface. In this case it follows from (6.4) and (2) that

$$\cos \beta(\eta) = \frac{1}{n[X_0(\eta)]} \frac{ds^i}{d\eta} = \frac{1}{n[X_0(\eta)]} \nabla s^i \cdot \frac{dX_0}{d\eta} = \cos \alpha(\eta). \tag{4}$$

Here $\beta(\eta)$ is the semi-angle of the cone of diffracted rays emanating from the point $X_0(\eta)$ of the edge and $\alpha(\eta)$ is the angle between the incident ray and the edge at that point. Since both angles lie between zero and π it follows from (4) that they are equal. Thus we have obtained the <u>special law of edge diffraction</u>: The angle of diffraction is equal to the angle of incidence. Incident and diffracted rays in the neighborhood of a typical point on an edge are illustrated in the following figure.



If the edge lies on an interface there may be two or more wedge-shaped regions in the neighborhood of the edge with values of $n(X)$ continuous in each wedge but discontinuous across surfaces radiating from the edge and separating the wedges. In this case (6.4) and (2) yield the <u>general law of edge diffraction</u>:

$$n^d \cos \alpha^d = n^i \cos \alpha^i. \tag{5}$$

Here α^d and α^i are the angle of diffraction and the angle of incidence, and n^d and n^i are the values of the index of refraction in the regions containing the diffracted and incident ray, at the point of diffraction.

For vertices, which are secondary point sources, (2) has no special consequences. Diffracted rays emanate from the vertex in all outward directions in the domain of the problem.

Once the diffracted rays and the values of z_m^d on M are determined, the $z_m^d(X)$ and hence the diffracted wave u^d follow immediately from the formulas of section A7.

A15. Diffraction by edges: examples.*

To illustrate the foregoing theory we shall consider some problems with edges on the boundary of a medium with index of refraction $n = 1$. We begin with the case in which the edge is a straight line and the incident rays all lie in planes normal to the edge. Then the diffracted rays are also normal to the edge and emanate from it in all directions. Thus it suffices to consider all the rays in one plane normal to the edge. If r denotes distance from the edge, then the phase s^d of the diffracted wave is equal to $s^i + r$, where s^i is the phase of the incident wave at the edge. The edge lies on an incident wave front, hence s^i is constant on the edge. Since the diffracted wave is cylindrical, $z^d(r)$ is given by

$$z^d(r) = \bar{z}^d(0) r^{-1/2} = (d) z^i \, r^{-1/2}. \tag{1}$$

Here (d) denotes a diffraction coefficient and z^i is evaluated at the edge. Thus the leading term of the diffracted wave is given by

$$u^d \sim (d) z^i r^{-1/2} \, e^{ik(r+s^i)} \sim (d) u_g^i \, r^{-1/2} \, e^{ikr}. \tag{2}$$

In (2), $u_g^i = e^{iks^i} z^i$ denotes the "geometrical optics term", i.e., the leading term, of the incident wave, evaluated at the edge.

Let us compare our result (2) with Sommerfeld's exact solution [42] for diffraction of a plane wave by a half-plane. That result consists of the incident and reflected waves of geometrical optics plus a third, or "diffracted" term. When the third term is asymptotically expanded for large values of kr it agrees perfectly with (2), provided that

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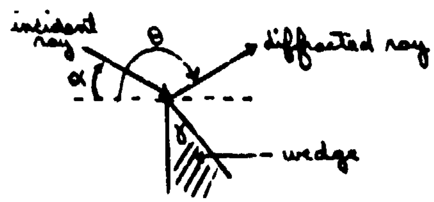
*Most of the material in this section is adapted from [16].

$$(d) = -\frac{e^{i\frac{\pi}{4}}}{2(2\pi k)^{1/2}\sin\beta}\left[\sec\frac{1}{2}(\theta-\alpha) \pm \csc\frac{1}{2}(\theta+\alpha)\right]. \tag{3}$$

Here β is the angle of incidence (or angle of diffraction) which is $\pi/2$ in the case we are considering. The angles between the incident and diffracted rays and the normal to the half-plane are α and θ respectively. They are illustrated in the following figure. (The wedge is a half-plane when $\gamma = 0$).



The upper sign in (3) applies when the boundary condition on the half-plane is $u = 0$, while the lower sign applies if it is $\frac{\partial u}{\partial v} = 0$.

The agreement between (2) and the exact solution of the canonical problem (i.e., the Sommerfeld problem) is a confirmation of our theory and also determines the edge diffraction coefficient (d). Similar agreement occurs for oblique incidence on a half-plane when (2) is replaced by the appropriate expression and the denominator $\sin\beta$ is included in (3). In this case θ and α are defined as above after first projecting the rays into the plane normal to the edge. In case the half-plane is replaced by a wedge of angle

$$\gamma = (2-q)\pi \tag{4}$$

comparison of (2), and its modified form for $\beta \neq \pi/2$, with Sommerfeld's exact solution for a wedge, yields agreement when

$$(d) = \frac{e^{i\frac{\pi}{4}}\sin\frac{\pi}{q}}{q(2\pi k)^{1/2}\sin\beta}\left[\left(\cos\frac{\pi}{q} - \cos\frac{\theta-\alpha}{q}\right)^{-1} \mp \left(\cos\frac{\pi}{q} - \cos\frac{\theta+\alpha}{q}\right)^{-1}\right]. \tag{5}$$

For $q = 2$, the wedge becomes a half-plane and (5) reduces to (3).

We shall now apply (2) and (3) to determine the field diffracted through an infinitely long slit of width 2a in a thin screen. For simplicity we shall assume that the incident field is a plane wave propagating in a direction normal to the edges of the slit. Then we can confine our attention to a plane normal to the edges. In this plane let the screen lie on the y axis of a rectangular co-ordinate system with the edges of the slit at x = 0 and y = $\pm$ a. Let the incident field be the plane wave

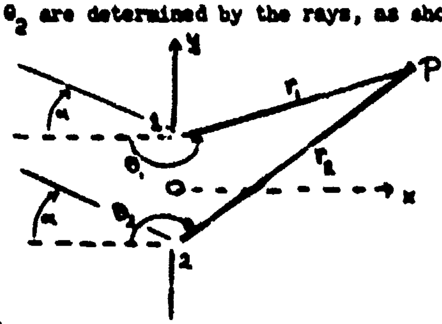
$$u^i = e^{ik(x \cos \alpha - y \sin \alpha)}. \tag{6}$$

Two singly-diffracted rays, one from each edge, pass through any point P. Thus the leading term of the singly-diffracted field at P, $u_1^d(P)$ is the sum of two terms,

$$u_1^d(P) \sim - \frac{e^{ik(r_1 - a \sin \alpha) + i\frac{\pi}{4}}}{2(2\pi k r_1)^{1/2}} \left[\sec \tfrac{1}{2}(\theta_1 + \alpha) \pm \csc \tfrac{1}{2}(\theta_1 - \alpha)\right]$$

$$- \frac{e^{ik(r_2 + a \sin \alpha) + i\frac{\pi}{4}}}{2(2\pi k r_2)^{1/2}} \left[\sec \tfrac{1}{2}(\theta_2 - \alpha) \pm \csc \tfrac{1}{2}(\theta_2 + \alpha)\right]. \tag{7}$$

In (7), r_1 and r_2 denote the distances from P to the upper and lower edges, and the angles θ_1 and θ_2 are determined by the rays, as shown in the figure.



The result (7) can be improved by adding to it the leading term of the doubly-diffracted field $u_2^d(P)$ which consists of the sum of two terms corresponding to the two doubly-diffracted rays passing through P. Each of these rays begins at one edge of the slit, is diffracted from the other edge, and then passes through P. To find the corresponding waves it is necessary to treat the two singly-diffracted waves emanating from the two edges as new incident waves on the opposite edges of the slit, and then to apply (2) and (3) with $\alpha = \pi/2$. The computation is straightforward when the boundary condition is $u = 0$:

From (7) we see that at edge $\binom{1}{2}$ the leading term of the singly-diffracted wave emanating from edge $\binom{2}{1}$ is given by

$$u\binom{1}{2} \sim -\frac{e^{ika(2 \pm \sin\alpha) + i\pi/4}}{4(\pi ka)^{1/2}} \left[\sec\tfrac{1}{2}\left(\tfrac{\pi}{2} \mp \alpha\right) + \csc\tfrac{1}{2}\left(\tfrac{\pi}{2} \pm \alpha\right)\right]. \qquad (8)$$

Here we have used the upper sign in each term of (7) corresponding to the boundary condition, $u = 0$, and have chosen the appropriate values of r_j and θ_j. (8) is easily simplified to yield

$$u(j) \sim -\frac{e^{ika[2-(-1)^j \sin\alpha] + i\pi/4}}{2(\pi ka)^{1/2}} \sec\tfrac{1}{2}\left[\tfrac{\pi}{2} + (-1)^j \alpha\right]. \qquad (9)$$

From (2), the leading term of the doubly-diffracted field at P is given by

$$u_2^d(P) \sim \sum_j (d)\, u(j)\, r_j^{-\frac{1}{2}}\, e^{ikr_j}, \qquad (10)$$

where

$$(d) = -\frac{e^{i\pi/4}}{2(2\pi k)^{1/2}} \left[\sec\tfrac{1}{2}\left(\theta_j - \tfrac{\pi}{2}\right) + \csc\tfrac{1}{2}\left(\theta_j + \tfrac{\pi}{2}\right)\right] = -\frac{e^{i\pi/4}}{(2\pi k)^{1/2}} \sec\tfrac{1}{2}\left(\theta_j - \tfrac{\pi}{2}\right). \qquad (11)$$

By inserting (11) and (9) in (10) we obtain

$$u_2^d(P) \sim \sum_{j=1}^{2} \frac{i e^{ika[2-(-1)^j \sin\alpha] + ikr_j}}{2\pi k(2ar_j)^{1/2}} \sec\tfrac{1}{2}(\theta_j - \tfrac{\pi}{2}) \sec \tfrac{1}{2}[\tfrac{\pi}{2} + (-1)^j \alpha]. \tag{12}$$

We note that $u_1^d(P)$ is of order $k^{-\frac{1}{2}}$ and $u_2^d(P)$ is of order k^{-1}. Clearly $u_j^d(P)$ will be of order $k^{-j/2}$. Here $u_j^d(P)$ is the leading term of the field corresponding to the j-tuply diffracted rays. It too consists of a sum of two waves. Since u_3^d is of order $k^{-3/2}$ it is of the same order as the second term in each of the singly-diffracted waves. We have not computed these terms because so far we are unable to compute the amplitude coefficients. If we denote by u_g the geometrical optics field (i.e., the incident and reflected fields) we may write the solution of the problem of diffraction by an infinite slit, with boundary condition $u = 0$ in the form

$$u \sim u_g + u_1^d + u_2^d + O(k^{-\frac{3}{2}}). \tag{13}$$

Although the leading terms of the remaining multiply-diffracted waves are no larger than terms omitted in (13) it is interesting to note that they are easily computed. In fact the resulting series is a geometric series, hence is easily summed (See [20]).

We have not computed the doubly-diffracted wave for the slit problem with the boundary condition $\frac{\partial u}{\partial n} = 0$, corresponding to the lower sign in (3).

In this case (d) vanishes when $\alpha = \pi/2$. This is to be expected, for if a plane wave travels toward a half-plane in a direction parallel to the plane, the incident plane wave itself satisfies the boundary condition condition $\frac{\partial u}{\partial \nu} = 0$ and no diffracted wave is produced. If an arbitrary wave u^i is incident in the same direction, we assume that the diffracted wave is proportional to $\frac{\partial u^i}{\partial \nu}$, the normal derivative of the incident wave at the edge. The proportionality factor is a new diffraction coefficient which can be obtained by solving an appropriate canonical problem. This new coefficient and its application are given in [16].

Thus far we have considered only problems with straight edges. For a curved diffracting edge, let r denote distance along a diffracted ray from the edge. Then the leading term of the diffracted wave is given by

$$u^d \sim e^{iks^d} z_0^d \; . \tag{14}$$

Here

$$s^d = s^i + r, \tag{15}$$

and, from (7.9),

$$z_0^d(r) = \tilde{z}_0^d(0) \left[r\left(1+\frac{r}{\rho_1}\right)\right]^{-\frac{1}{2}} = (d) z_0^i \left[r\left(1+\frac{r}{\rho_1}\right)\right]^{-\frac{1}{2}}. \tag{16}$$

In (15) and (16), s^i and z_0^i denote the phase and amplitude of the incident wave at the point of diffraction. ρ_1 is given by (7.17). If the diffracting edge is the edge of a thin screen and the boundary condition on the screen is $u=0$ or $\frac{\partial u}{\partial n} = 0$, then (d) is given by (3). If, in a neighborhood of the point of diffraction, the boundary is locally wedge-shaped, then (d) is given by (5).

To illustrate diffraction by curved edges, we consider the problem of a plane wave, $u^i = e^{ikx}$, normally incident upon a plane screen, $x = 0$, containing a circular aperture of radius a. The geometry can be visualized with the aid of the second figure of this section. Then $\alpha = 0$, and two singly-diffracted rays pass through every point P. They come from the nearest and farthest points on the edge. The angle of incidence β is everywhere $\pi/2$ and the radius of curvature ρ of the edge is a. For both diffracted rays the angle δ between the ray and the normal to the edge (which lies in the plane of the aperture) satisfies $\delta = \theta - \pi/2$. Hence (7.17) becomes $\rho_1 = -a/\sin\theta$. Then (3), (7.9) and (14.3) yield the "singly diffracted field",

$$u_1^d(P) \sim -\sum_{j=1}^{2} \frac{e^{ikr_j + i\pi/4}}{2(2\pi k)^{1/2}} \left[\sec\tfrac{1}{2}\theta_j \pm \csc\tfrac{1}{2}\theta_j\right] \left[r_j(1-a^{-1}r_j \sin\theta_j)\right]^{-\frac{1}{2}} \tag{17}$$

Here we have added the contribution corresponding to the two singly-diffracted rays passing through P. On the x-axis, $r_j \sin\theta_j = a$, hence the last factor in (17) is infinite. This occurs because the axis is a caustic of the diffracted waves. Since the exact solution of the problem is everywhere finite, a better asymptotic expansion is required in the neighborhood of the axis. Such expansions are discussed in [20]. If the aperture, instead of being circular, is formed by a smooth convex curve, (17) is essentially unchanged. Again two singly-diffracted rays pass through each point P, emanating from the nearest and farthest points on the edge of the aperture. The singly-diffracted field will be given by (17) if we interpret the angles and distances in the obvious way. In each term, a must be replaced by a_j, the radius of curvature of the edge at the point of diffraction.

If a plane wave is normally incident upon a plane screen containing an aperture, the edge of which is an arbitrary regular curve, the diffracted rays emanating from each point of the edge lie in a plane perpendicular to the edge. The envelope of these planes is a cylinder with generators normal to the plane of the screen. This cylinder is, of course, a caustic surface of the singly-diffracted wave. (The other caustic is the edge itself.). The cross-section of the cylinder formed by its intersection with the plane of the screen is a curve. This curve is the envelope of the normals to the edge, i.e., the <u>evolute</u> of the edge. Thus on every plane parallel to the screen the caustic intersects the plane in the evolute curve, and one would expect to find corresponding bright lines in the diffraction patterns formed on such planes. These bright lines have been observed and constitute an interesting experimental confirmation of our theory. When the evolute lies within the aperture curve, the lines are masked by the presence of the incident wave passing through the aperture. In such cases they are more easily observed when the aperture is replaced by the complimentary screen, e.g., when the circular aperture is replaced by a circular disk.

A16. <u>Expansions containing exponential decay factors and fractional powers of k.</u>

The asymptotic solutions of problems for the reduced wave equation which we have considered so far have been based on an expansion of the form (5.1). However, more general types of expansions have been discovered by asymptotically expanding exact solutions of the reduced wave equation,

$$\nabla^2 u + k^2 u = 0, \qquad (1)$$

for a homogeneous medium. In [7], Friedlander and Keller have made a systematic study of asymptotic solutions of (1) of the form

$$u \sim \exp(iks(X) - k^{\alpha} p(X)) \sum_{m=0}^{\infty} \frac{z_m(X)}{k^{\lambda_m}}. \qquad (2)$$

Here α and λ_m are real numbers and $\lambda_{m+1} > \lambda_m$. Although formal solutions of the type (2) exist for all values of α only the values $\alpha = 0$ and $\alpha = \frac{1}{3}$ have occurred in actual problems. Since the case $\alpha = 0$ reduces to the expansion (5.1), we may restrict our attention here to the case $\alpha = \frac{1}{3}$. Thus we consider asymptotic solutions of the reduced wave equation (1.5) of the form (1.8) where

$$z = e^{-k^{1/3} p(X)} v. \qquad (3)$$

Since we have shown that z satisfies (1.9) we may insert (3) in (1.9) to obtain

$$-k^2[(\nabla s)^2 - n^2]v - 2ik^{4/3} v \nabla s \cdot \nabla p + ik[2\nabla s \cdot \nabla v + v\Delta s]$$

$$+ k^{2/3} (\nabla p)^2 v - k^{1/3}(2\nabla p \cdot \nabla v + v\Delta p) + \Delta v = 0. \qquad (4)$$

From the form of (4) it is clear that we may expect that w will have an expansion in reciprocal powers of $k^{1/3}$. Thus we set

$$w = \sum_{m=0}^{\infty} z_m(x) k^{-\frac{m}{3}} = \sum_{m} z_m(x) k^{-\frac{m}{3}}, \tag{5}$$

where $z_m = 0$ for $m = -1,-2,\ldots$. If we insert (5) in (4) and collect like powers of $k^{1/3}$ we obtain the equations

$$(\nabla s)^2 = n^2, \tag{6}$$

$$\nabla s \cdot \nabla p = 0, \tag{7}$$

and

$$2\nabla z_m \cdot \nabla s + z_m \Delta s = r_m. \tag{8}$$

Here

$$r_m = i z_{m-1}(\nabla p)^2 - i[2\nabla z_{m-2} \cdot \nabla p + z_{m-2}\Delta p] + i\,\Delta z_{m-3}. \tag{9}$$

We note that (6) is the familiar eiconal equation. It follows that the main features of our earlier expansion, i.e., the rays and wave-fronts, are preserved in the new expansion. (7) merely asserts that the surfaces p = const are orthogonal to the wave-fronts s = const., i.e.,

$$p = \text{const. on each ray.} \tag{10}$$

For $m = 0$, $r_m = 0$ and (8) is identical to the zero-order transport equation. For arbitrary m (8) can be written in the form

$$2n\frac{dz_m}{d\sigma} + z_m \Delta s = r_m. \tag{11}$$

Here σ denotes arclength along a ray. By comparison with (3.14) we easily obtain the solution of the ordinary differential equation (11),

$$z_m(\sigma) = z_m(\sigma_0)\left[\frac{\xi(\sigma_0)n(\sigma_0)}{\xi(\sigma)\,n(\sigma)}\right]^{1/2} + \frac{1}{2}\int_{\sigma_0}^{\sigma}\left[\frac{\xi(\sigma')n(\sigma')}{\xi(\sigma)n(\sigma)}\right]^{1/2}\frac{r_m(\sigma')}{n(\sigma')}\,d\sigma'. \tag{12}$$

For a homogeneous medium, n = const., and (12) becomes (See section A4)

$$z_m(\sigma) = z_m(\sigma_0)\left[\frac{(\rho_1+\sigma_0)(\rho_2+\sigma_0)}{(\rho_1+\sigma)(\rho_2+\sigma)}\right]^{1/2} + \frac{1}{2n}\int_{\sigma_0}^{\sigma}\left[\frac{(\rho_1+\sigma')(\rho_2+\sigma')}{(\rho_1+\sigma)(\rho_2+\sigma)}\right]^{1/2} r_m(\sigma')d\sigma'. \tag{13}$$

Our new expansion, which takes the form

$$u \sim \exp\left(iks(X) - k^{1/3}p(X)\right)\sum_{m=0}^{\infty} z_m(X)k^{-\frac{m}{3}}, \tag{14}$$

will also be called a "wave". It will be required shortly in our discussion of diffraction by smooth bodies.

A17. The surface eiconal equation and surface rays.

In preparation for our study of diffraction by smooth bodies we consider now the initial value problem for the eiconal equation on a surface. We are concerned with a function s defined only on a surface S, and with initial values prescribed on a curve which lies on that surface. Let $X = X(\tau_1, \tau_2)$ be a parametric equation for the regular surface, S. Following the customary notation of the differential geometry of surfaces, we introduce the surface tangent vectors

$$X_1 = \frac{\partial X}{\partial \tau_1} \;, \quad X_2 = \frac{\partial X}{\partial \tau_2} \;, \tag{1}$$

and the metric coefficients

$$g_{ij} = X_i \cdot X_j ; \; i,j = 1,2. \tag{2}$$

We also introduce the inverse (g^{ij}) of the matrix (g_{ij}). Then, of course

$$g^{ki} \, g_{ij} = \delta_{kj} \, . \tag{3}$$

In (3) and subsequent equations we employ the summation convention for repeated indices over the values 1,2. δ_{kj} is the Kroenecker symbol.

For any function $f(\tau_1,\tau_2)$ of the surface parameters, let $f_i = \frac{\partial f}{\partial \tau_i}$. The surface gradient of f is defined by

$$\tilde{\nabla} f = g^{ki} \, f_i \, X_k. \tag{4}$$

To see that (4) agrees with the usual definitions of the gradient we set $dX = X_\nu d\tau_\nu$ and observe that

$$\tilde{\nabla}f \cdot dX = g^{ki} f_i X_k \cdot X_\nu d\tau_\nu = f_i g^{ki} g_{k\nu} d\tau_\nu = f_i \delta_{i\nu} d\tau_\nu = f_\nu d\tau_\nu = df. \tag{5}$$

It follows now that $(\tilde{\nabla}s)^2 = g^{ij} s_i s_j$. Thus if we introduce the index of refraction $n(\tau_1, \tau_2) = n[X(\tau_1, \tau_2)]$, the surface eiconal equation can be written in the equivalent forms,

$$(\tilde{\nabla}s)^2 = n^2; \quad h(s_1, s_2, \tau_1, \tau_2) = g^{ij}(\tau_1, \tau_2) s_i s_j - n^2(\tau_1, \tau_2) = 0. \tag{6}$$

In order to solve the first order partial differential equation (6), we introduce the characteristic curves $[\tau_1(\sigma), \tau_2(\sigma)]$ which are determined by the solutions of the characteristic equations (Hamilton's equations)

$$\dot{\tau}_i = \frac{\lambda}{2} \frac{\partial h}{\partial s_i} = \lambda g^{ij} s_j, \tag{7}$$

$$\dot{s}_i = -\frac{\lambda}{2} \frac{\partial h}{\partial \tau_i} = -\frac{\lambda}{2} \left[(g^{kj})_i s_k s_j - (n^2)_i \right]. \tag{8}$$

Here the dot denotes differentiation with respect to the parameter, σ. (6), (7), and (8) imply that

$$\dot{X}^2 = (X_i \dot{\tau}_i)^2 = g_{ik} \dot{\tau}_i \dot{\tau}_k = \lambda^2 g_{ik} g^{ij} s_j g^{k\nu} s_\nu = \lambda^2 \delta_{kj} s_j g^{k\nu} s_\nu = \lambda^2 g^{k\nu} s_k s_\nu \tag{9}$$

$$= \lambda^2 n^2.$$

Hence we may identify the parameter σ with arclength along the surface curves $X = X(\sigma) = X[\tau_1(\sigma),\ \tau_2(\sigma)]$ by setting

$$\lambda = \frac{1}{n} \quad . \tag{10}$$

These curves will be called <u>surface rays</u>. We note that the equation

$$\dot{X} = X_i\ \dot{\tau}_i = \lambda g^{ij} s_j X_i = \frac{1}{n}\tilde{\nabla} s \tag{11}$$

implies that the surface rays are everywhere orthogonal to the <u>surface wave-fronts</u> $s(\tau_1, \tau_2) = \text{const}$. From (7), (6) and (10)

$$\dot{s} = s_i \dot{\tau}_i = \lambda g^{ij}\ s_i s_j = \lambda n^2 = n, \tag{12}$$

hence

$$s[X(\sigma)] = s[X(\sigma_0)] + \int_{\sigma_0}^{\sigma} n[X(\sigma')]\,d\sigma'. \tag{13}$$

(13) provides the solution of the surface eiconal equation (6), once initial values are specified .

We assume that the initial values are given on a curve on the surface, $(\tau_1, \tau_2) = [\tau_1(\eta),\ \tau_2(\eta)]$, where η is an arclength parameter. Thus the initial data take the form

$$s[X(\eta)] = s^0(\eta). \tag{14}$$

Here $s^0(\eta)$ is a given function. Differentiation of (14) yields

$$\tilde{\nabla} s \cdot \frac{dX}{d\eta} = \frac{ds^0}{d\eta}\,, \tag{15}$$

or

$$\cos\beta = \frac{1}{n}\,\frac{ds^{o}}{d\eta}\,. \tag{16}$$

Here β is the angle between a surface ray and the initial curve. If we assume that $-1 < \frac{1}{n}\,\frac{ds^{o}}{d\eta} < 1$ then (16) implies that at every point on the initial curve, one surface ray is outgoing from that curve on each side of it. These surface rays, together with (13) provide the outgoing solution of the initial value problem for the surface eiconal equation.

We recall that the rays associated with the eiconal equation (1.14) become straight lines in the case n = const. These straight lines are, of course, geodesics, or shortest paths between two points in space. We will now prove that for the case n = const. the surface rays defined by (7) and (8) are geodesics of the surface S. The proof will occupy the remainder of this section.

If n = const. (7) and (8) take the form

$$\dot{y}_i = \frac{1}{n}\,g^{ij}s_j\;,\qquad \dot{s}_j = -\frac{1}{2n}\,(g^{k\nu})_j\,s_k s_\nu\,. \tag{17}$$

We shall replace this system of four first-order ordinary differential equations by a system of two second-order equations. We first note that

$$\ddot{y}_r = \frac{1}{n}\,g^{rj}\dot{s}_j + \frac{1}{n}(g^{rj})_m s_j \dot{y}_m = -\frac{1}{2n^2}\,g^{rj}(g^{k\nu})_j s_k s_\nu + \frac{1}{n^2}(g^{r\nu})_m\,g^{mk}s_k s_\nu \tag{18}$$

and

$$\dot{r}_i \dot{r}_j = \frac{1}{n^2} g^{ik} s_k g^{j\nu} s_\nu . \tag{19}$$

Next we introduce the Christoffel symbol $\{{}^{j}{}_{r}{}^{i}\}$ defined by

$$\{{}^{j}{}_{r}{}^{i}\} = \frac{1}{2} g^{rm}((g_{jm})_i + (g_{im})_j - (g_{ji})_m), \tag{20}$$

and (19) and (20) yield

$$\{{}^{j}{}_{r}{}^{i}\} \dot{r}_i \dot{r}_j = \frac{1}{2n^2} g^{ik} g^{j\nu} g^{rm}((g_{jm})_i + (g_{im})_j - (g_{ji})_m) s_k s_\nu . \tag{21}$$

This equation can be simplified by using an identity obtained by differentiating (3),

$$(g^{ki})_\nu g_{ij} = -g^{ki}(g_{ij})_\nu . \tag{22}$$

The result is

$$\{{}^{j}{}_{r}{}^{i}\} \dot{r}_i \dot{r}_j = -\frac{1}{2n^2} (g^{ik} g^{rm} g_{jm} (g^{j\nu})_i + g^{j\nu} g^{rm} g_{im} (g^{ik})_j - g^{rm} g^{ik} g_{ji} (g^{j\nu})_m) s_k s_\nu$$

$$= -\frac{1}{2n^2} (g^{ik}(g^{r\nu})_i + g^{j\nu}(g^{rk})_j - g^{rm}(g^{k\nu})_m) s_k s_\nu . \tag{23}$$

From (18) and (23) we easily obtain the second order system of differential equations

$$\ddot{r}_r + \{{}^{j}{}_{r}{}^{i}\} \dot{r}_i \dot{r}_j = 0. \tag{24}$$

These are the differential equations of a surface geodesic.*

* See, e.g., Stoker, J.J., Differential Geometry, New York University Lecture notes, (1955), page V-5, equation V.4.

A 18. Diffraction by smooth objects*

In this section we shall derive a general formula for the diffracted wave which is produced when a wave u^i is incident on a smooth surface S in such a way that some of the incident rays are tangent to S along a curve C. In this case there is a shadow region which is not penetrated by any of the ordinary rays of geometrical optics. The shadow region is separated from the region reached by incident and reflected rays by a surface called the shadow boundary. The tangent rays, beyond their points of tangency, lie on the shadow boundary. For simplicity, we shall assume that S is a boundary rather than an interface. Thus we shall avoid the additional complications of transmitted waves. As in all of our considerations, the following construction will involve certain apparently aribtrary prescriptions. These prescriptions were discovered by examining the asymptotic expansion of exact solutions of boundary value problems for the reduced wave equation. They will be further verified by the boundary layer theory. However, general proofs of the validity of the formula have not yet been given.

In order to derive the formula for the diffracted wave u^d we first construct a surface wave (or creeping wave) u^c which is defined only on the surface S. The curve C acts as the (secondary) source of the surface wave, which is excited by the incident wave u^i. u^c is defined only on the "dark" side of C, i.e., on the portion of S adjacent to the shadow region. On this portion of S, the phase s^c of the surface wave satisfies the surface eiconal equation (17.6) with initial conditions given by

$$s^c = s^i \qquad \text{on } C. \tag{1}$$

*Much of the material in this section is adapted from [36].

It follows easily from (1) that at each point Q_1 on C the surface ray emanating from that point is tangent to the incident ray (which is tangent to S at Q_1). If P_1 is any other point on the surface ray emanating from Q_1, we see from (17.13) that

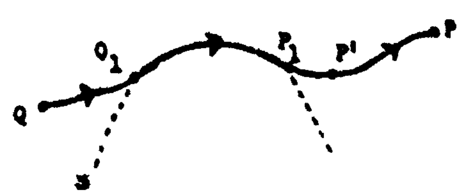
$$s^c(P_1) = s^i(Q_1) + \int_{Q_1}^{P_1} n d\sigma. \tag{2}$$

Here the variable of integration σ is arclength along the surface ray.

Before finding the amplitude of the surface wave we will begin the construction of the diffracted wave. The "dark" surface of S acts as the (secondary) source exciting u^d. The phase s^d of the diffracted wave satisfies the eiconal equation (1.14) with initial data given by

$$s^d = s^c \quad \text{on } S. \tag{3}$$

We see from section 6 that s^d is the solution of a "characteristic initial value problem" for the eiconal equation, and that at every point P_1 on S the diffracted ray emanating from P_1 is tangent not only to S but also to the surface ray passing through P_1. Portions of the incident, surface, and diffracted rays are sketched in the following figure.



The rays may be described as follows: The incident ray which is tangent to S at Q_1 splits into two branches. One branch (not shown in the figure) continues along the shadow boundary; the other branch is the surface ray. At every point P_1 (only one point is shown) on its path the surface ray splits into two branches. One branch(not shown) continues along the surface; the other branch is the diffracted ray emanating from P_1. From (6.12) the phase of the diffracted wave is given by

$$s^d(P) = s^c(P_1) + \int_{P_1}^{P} n d\sigma = s^i(Q_1) + \int_{Q_1}^{P_1} n d\sigma + \int_{P_1}^{P} n d\sigma. \qquad (4)$$

The leading term of the surface wave is given by $u^c \sim e^{iks^c} z_o^c$. In order to construct the amplitude z_o^c, we consider the width $dw(\sigma)$ of an infinitesimal strip of surface rays at the point σ on a given surface ray. The "energy flux" through such a strip is proportional to $n(\sigma)[z_o^c(\sigma)]^2 dw(\sigma)$. At the point $\sigma + d\sigma$ we assume that the flux is smaller due to energy lost to the diffracted rays which emanate from the surface rays in the interval $d\sigma$, and that the energy loss is proportional to $n(z_o^c)^2$ and to the area element $dw d\sigma$. Thus

$$d[n(z_o^c)^2 dw] = -2\alpha\, n(z_o^c)^2\, dw d\sigma. \qquad (5)$$

The <u>decay exponent</u> $\alpha(\sigma)$ depends on local properties of the surface, the medium, and the field. Integration of (5) yields

$$z_o^c(\sigma) = z_o^c(0)\left[\frac{n(0)}{n(\sigma)}\frac{dw(0)}{dw(\sigma)}\right]^{\frac{1}{2}} \exp\left(-\int_0^{\sigma} \alpha(\sigma')d\sigma'\right). \qquad (6)$$

We now assume that the amplitude of the surface wave at Q_1 is proportional to the amplitude of the incident wave at that point,

$$z_o^c(Q_1) = d(Q_1) z_o^i(Q_1). \qquad (7)$$

Here $d(Q_1)$ is a diffraction coefficient. From (7.4) we obtain the formula for the amplitude $z_o^d(s)$ of the diffracted wave at a distance s along the diffracted ray from the point P_1,

$$z_o^d(s) = z_o^d(0)\left[\rho_2 \sin\gamma \frac{d\theta_1 d\theta_2}{d\rho(s)} \frac{n(0)}{n(s)}\right]^{\frac{1}{2}}. \qquad (8)$$

The quantities ρ_2, γ, $d\theta_1$, and $d\theta_2$ are defined in section 7. We assume that the amplitude of the diffracted wave is proportional to that of the surface wave at the point P_1,*

$$z_o^d(o) = k^{-\frac{1}{2}} d(P_1) z_o^c(P_1) . \qquad (9)$$

The diffraction coefficients $d(Q_1)$ and $d(P_1)$ are assumed to be the same function of the properties of the surface, medium, and the field at the respective points Q_1 and P_1. This assumption is based on the reciprocity principle--a source at Q produces the same field at P that a source at P produces at Q. We shall see that the values of the diffraction coefficients and the decay exponent depend on the boundary condition.

From (6 - 9) we now obtain

$$z_o^d(P) = k^{-\frac{1}{2}} d(P_1) d(Q_1) z_o^i(Q_1) \left[\frac{d\omega(Q_1)}{d\omega(P_1)} \; \frac{n(Q_1)}{n(P)} \rho_2 \sin \gamma \; \frac{d\theta_1 d\theta_2}{da(P)} \right]^{\frac{1}{2}} \exp\left\{ - \int_{Q_1}^{P_1} \alpha(s) ds \right\} . \qquad (10)$$

Then the leading term of the diffracted field is given by (4), (10), and

$$u^d(P) \sim e^{iks^d(P)} z_o^d(P) . \qquad (11)$$

These equations were derived using the surface wave $u^c \sim e^{iks^c} z_o^c$, which no longer appears in the result. The surface wave is not an asymptotic representation of the true solution at the boundary. (This is especially clear if the boundary condition is $u = 0$.) It is merely a convenient intermediate step for the description of the diffracted wave. The latter is singular at the boundary because the diffracted rays have a caustic there.

Our construction so far is not quite complete. Actually the diffracted wave consists of a number of modes u_j^d; $j = 1, 2, \ldots$ of which we have constructed only one in (11). Each mode has its own diffraction coefficient d_j and decay exponent α_j. Thus the diffracted field is given by

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*The factor $k^{-\frac{1}{2}}$ in (9) is required in order that $d(P_1)$ should be dimensionless. (See Equation (7.20).)

$$u^d(P) \sim z_0^i(Q_1) \exp\left\{ik\left[s^i(Q_1) + \int_{Q_1}^{P_1} n\,d\sigma + \int_{P_1}^{P} n\,d\sigma\right]\right\} \left[\frac{d\omega(Q_1)}{d\omega(P_1)} \frac{n(Q_1)}{n(P)} \rho_2 \sin\gamma\right.$$

$$\left. \times \frac{d\Theta_1 d\Theta_2}{da(P)}\right]^{\frac{1}{2}} \sum_j k^{-\frac{1}{2}} d_j(P_1) d_j(Q_1) \exp\left\{-\int_{Q_1}^{P_1} \alpha_j(\sigma)\,d\sigma\right\} . \tag{12}$$

For the case of a homogeneous medium, n is constant and (7.21) and (12) yield

$$u^d(P) \sim z_0^i(Q_1) \exp\left\{ik\left[s^i(Q_1) + n\tau + n\sigma\right]\right\} \left[\frac{d\omega(Q_1)}{d\omega(P_1)} \frac{\rho_2}{\sigma(\rho_2+\sigma)}\right]^{\frac{1}{2}}$$

$$\times \sum_j k^{-\frac{1}{2}} d_j(P_1) d_j(Q_1) \exp\left\{-\int_{Q_1}^{P_1} \alpha_j(\sigma)\,d\sigma\right\} . \tag{13}$$

Here σ is the distance from P_1 to P and τ is the distance from Q_1 to P_1 along the surface geodesic. Values of d_j and α_j will be given in (19.13) and (19.14) of Section 19.

If we use (7.24) instead of (7.19) in deriving (8) then (12) takes the form

$$u^d(P) \sim z_0^i(Q_1) \exp\left\{ik\left[s^i(Q_1) + \int_{Q_1}^{P_1} n\,d\sigma + \int_{P_1}^{P} n\,d\sigma\right]\right\} \left[\frac{d\omega(Q_1)}{d\omega(P_1)} \frac{n(Q_1)}{n(P)} \frac{d\tilde{a}(P_1)}{da(P)}\right]^{\frac{1}{2}}$$

$$\times \sum_j k^{-\frac{1}{2}} d_j(P_1) d_j(Q_1) \exp\left\{-\int_{Q_1}^{P_1} \alpha_j(\sigma)\,d\sigma\right\} . \tag{14}$$

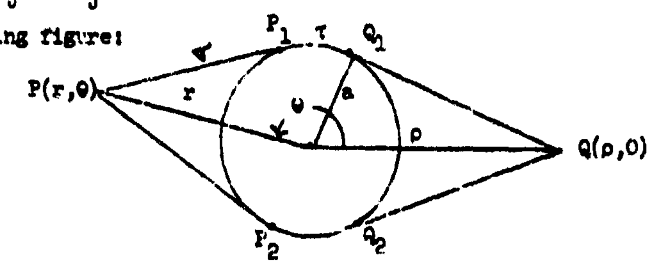
We see from (7.23) that

$$\frac{d\tilde{a}(P_1)}{da(P)} = \lim_{P' \to P_1} \frac{da(P')}{\sigma_0\, da(P)} . \tag{15}$$

Here P' is a point on the diffracted ray joining P_1 and P and σ_0 denotes the distance from P_1 to P' along this ray.

A 19. Diffraction by a circular cylinder*

From (18.7) and (18.9) we see that the diffraction coefficient d_j is dimensionless and (18.5) shows that the decay exponent α_j has the dimension of a reciprocal length. The diffraction coefficient must depend on k because we expect it to vanish for $k \to \infty$. Thus d_j must be a function of ka where a is a length. We assume that for a homogeneous medium a is the radius of curvature of the normal section of S in the ray direction. We also assume that α_j depends only on k and a. Then d_j and α_j can be obtained from the asymptotic expansion of the exact solution of a problem with some simple surface S. In this section we shall find the expression for the field produced by a line source which is parallel to a circular cylinder of radius a in a medium with index of refraction n = 1. Comparison with the exact solution will yield the coefficients d_j and α_j. The problem, which is two-dimensional, is illustrated in the following figure:



Let r' denote distance from the source point Q, which is located at the point with polar co-ordinates $(\rho, 0)$. We take the incident wave, produced by this source to be (compare section 9)

$$u^i = \frac{i}{4} H_o^{(1)}(kr') \sim \frac{e^{-i(\pi/4)+ikr'}}{2\sqrt{2\pi kr'}} . \qquad (1)$$

---

* The material in this section is based on [29].

The surface rays, which are geodesics on the cylinder are clearly arcs of the circles which generate the cylinder, and in (18.13) it is clear that

$\frac{dw(Q_1)}{dw(P_1)} = 1$, and $\rho_2 = \infty$. The assumptions made above imply that $\alpha_j(\sigma) = \text{const.}$ and $d_j(P_1) = d_j(Q_1)$. Furthermore, since Q_1 is a point of tangency of a ray from Q, we see that at Q_1, $r' = (\rho^2-a^2)^{1/2}$. Similarly $\sigma = (r^2-a^2)^{1/2}$. Thus (18.13) yields

$$u^d(r,\theta) \sim \left[8\pi k^2(r^2-a^2)^{1/2}(\rho^2-a^2)^{1/2}\right]^{-1/2} \exp\left\{ik\left[(\rho^2-a^2)^{1/2}+(r^2-a^2)^{1/2}\right]+i\pi/4\right\}$$

$$\times \sum_j d_j^2 \exp\left((ik-\alpha_j)\,\tau\right). \qquad (2)$$

Equation (2) gives the field on a ray from Q to P having an arc of length τ on the cylinder. For the ray $Q\,Q_1P_1P$, $\tau = \tau_o$ where

$$\tau_o = a\theta - a\cos^{-1}(a/\rho) - a\cos^{-1}(a/r). \qquad (3)$$

But all rays which are tangent at Q_1, encircle the cylinder n times, and leave at P_1, also contribute to the diffracted field. For these rays, $\tau = \tau_n$ where

$$\tau_n = \tau_o + 2n\pi a. \qquad (4)$$

We note that

$$\sum_{n=0}^{\infty} e^{(ik-\alpha_j)\tau_n} = e^{(ik-\alpha_j)\tau_o}\left[1-e^{2\pi a(ik-\alpha_j)}\right]^{-1}. \qquad (5)$$

Therefore we may insert (4) in (2) and sum over n. This yields the field contribution

$$u_1^d(r,\theta) \sim (8\pi)^{-\frac{1}{2}}k^{-1}(r^2-a^2)^{-1/4}(\rho^2-a^2)^{-1/4} \exp(ik[(\rho^2-a^2)^{1/2}+(r^2-a^2)^{1/2}]+i\pi/4)$$

$$\times \sum_j d_j{}^2 \exp\{(ik-\alpha_j)\,\tau\} \; [1-\exp(2\pi a(ik-\alpha_j))]^{-1} \quad . \tag{6}$$

At every point P there is also a contribution u_2^d corresponding to rays which encircle the cylinder n times in the opposite direction and leave at P_2. u_2^d can be obtained by replacing θ by $2\pi-\theta$ in (6). Then the total diffracted field $u^d = u_1^d + u_2^d$ is given by

$$u^d(r,\theta) \sim (8\pi)^{-\frac{1}{2}}k^{-1}(r^2-a^2)^{-1/4} (\rho^2-a^2)^{-1/4} \exp(ik[(\rho^2-a^2)^{1/2}+(r^2-a^2)^{1/2}]+i\pi/4)$$

$$\times \sum_j d_j{}^2 \; [1-\exp(2\pi a(ik-\alpha_j))]^{-1}[\exp((ik-\alpha_j)a\theta)+\exp((ik-\alpha_j)a(2\pi-\theta))]$$

$$\times \exp\,(-(ik-\alpha_j)a[\cos^{-1}(a/\rho)+\cos^{-1}(a/r)]\,). \tag{7}$$

Except for the coefficients d_j and α_j, (7) is an explicit formula for the leading term of the diffracted field. In the shadow region, this is the only field. In the "lit region" it must be added to the incident and reflected fields. The coefficients d_j and α_j depend, of course, on the boundary condition specified on the cylinder $r = a$. We will take this condition to be the impedance boundary condition $\frac{\partial u}{\partial r} + ikzu = 0$. Here z is a constant.

In [29] the above problem is solved exactly by separation of variables and expanded asymptotically for large ka. The result agrees exactly with (7) if we set

$$\alpha_j = e^{-i\pi/6} \left(\frac{k}{6a^2}\right)^{1/3} q_j, \tag{8}$$

and

$$d_j = e^{5\pi i/8} (2\pi)^{1/4} \left\{ \frac{\pi}{6} e^{5\pi i/6} \left(\frac{ka}{6}\right)^{1/3} \left[\left\{ A'(q_j) \right\}^2 + q_j A^2(q_j)/3 \right]^{-1} \right\}^{1/2}. \tag{9}$$

Here q_j is the j^{th} solution of the equation

$$\frac{A'(q_j)}{A(q_j)} = e^{5\pi i/6} \left(\frac{ka}{6}\right)^{1/3} z, \tag{10}$$

and $A(x)$ is the Airy function

$$A(x) = \int_0^\infty \cos(\tau^3 - x\tau) d\tau. \tag{11}$$

$A'(x)$ denotes dA/dx.

Had we chosen a constant index of refraction n, other than n = 1 it is clear from the form of the reduced wave equation and the impedance boundary condition that (8-10) would be modified by replacing k by kn and z by z/n.

In order to determine the diffraction coefficient $d_j(x)$ and the decay exponent $\alpha_j(X)$ (at a point X on the boundary surface) for the case of an inhomogeneous medium, comparison with other exact solutions (See [36] and [37]) indicates that we must replace $1/a$ in (8-10) by the "relative curvature" $[1/a(X) + \kappa(X)]$ of the surface and the diffracted ray emanating from the point X. Here $a(X)$ is the radius of curvature of the normal section of S at the point X in the direction of the surface ray at that point, and $\kappa(x)$ is the curvature of the diffracted ray at X. We now make the replacements.

$$\frac{1}{a} \rightarrow \frac{1}{a(X)} + \kappa(X)\ , \quad k \rightarrow kn(X),\ z \rightarrow z/n(X) \tag{12}$$

in (8-10). The result, after some simplification of (9), is

$$\alpha_j(X) = e^{-i\pi/6} q_j \left[6^{-1}\, kn(X)\right]^{1/3} \left[a^{-1}(X) + \kappa(X)\right]^{2/3}, \tag{13}$$

and

$$d_j(X) = -e^{i\pi/24}\, 6^{-1/6}\, 2^{-1/4} \pi^{3/4} \left[kn(X)\right]^{1/6} \left[a^{-1}(X)+\kappa(X)\right]^{-1/6} \tag{14}$$

$$\times \left[q_j A^2(q_j) + 3\left\{A'(q_j)\right\}^2\right]^{-1/2}.$$

Here q_j is the j^{th} solution of the equation

$$\frac{A'(q_j)}{A(q_j)} = e^{5\pi i/6}\, zk^{1/3}\left\{6n^2(X)\left[a^{-1}(X)+\kappa(X)\right]\right\}^{-1/3}. \tag{15}$$

The boundary condition used in determining (13-15) is the impedance boundary condition

$$\frac{\partial u}{\partial \nu} + ikzu = 0 \text{ , on } S. \tag{16}$$

Here $\frac{\partial u}{\partial \nu}$ denotes the normal derivative. If z is not a constant on S it must be replaced by $z(X)$ in (15).

We see from (13) that α_j is of order $k^{1/3}$. Therefore the formulas (18.12-14) for the leading term of u^d agree with the general form of expansion studied in section 15.* Lower order terms in the expansion for u^d can, in principle, be obtained by boundary layer methods [41].

A20. Field of a line source in a plane stratified medium with a plane boundary**

Many interesting features of the foregoing theory can be illustrated by considering problems in which the index of refraction is constant on planes, i.e., a function of a single cartesian co-ordinate x, for in this case the ray equations can be integrated explicitly. We consider a problem with a plane boundary at $x = x_b$ and with an index of refraction $n(x)$ which increases monotonically for $x_b \leq x$. At $x = x_b$ we impose the impedance boundary condition, with constant z,

$$\frac{\partial u}{\partial x} + ikzu = 0 \text{ .} \tag{1}$$

A line source, perpendicular to the plane of the following figure, intersects this plane at the point $(x_o, 0)$.

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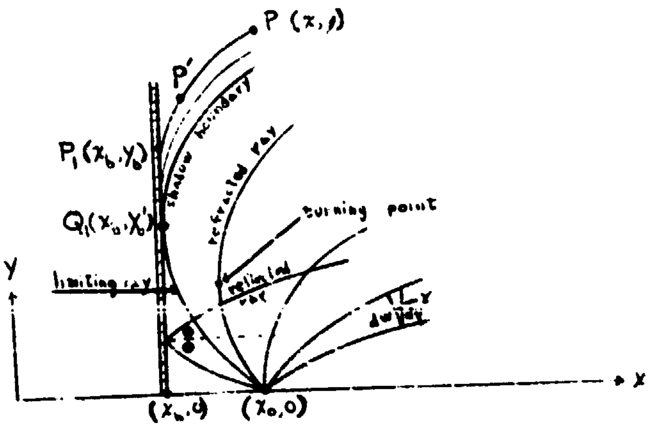
*This is strictly true only in the cases $z = 0$ and $z = \infty$. Otherwise we see from (15) that q^j is a function of k and the k dependence of α_j is obscured. If, however, we set $z = k^{-1/3}z_o$, where z_o is independent of k then q^j is independent of k and the above statement is true for all z_o.

**The material in this section is based on [26].



As in Section 9 we shall characterize the source by the inhomogeneous reduced wave equation $\nabla^2 u + k^2 n^2(x) u = -\delta(x-x_0)\delta(y)$. It suffices to confine our attention to the construction of the field in the upper half-plane $y \geq 0$.

If we set $\lambda = 1$ and denote the parameter in (2.4,5) by t, then the ray equations take the form

$$\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{n^2}{2}\right), \quad \frac{d^2y}{dt^2} = 0, \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = n^2. \tag{2}$$

To integrate (2) it is convenient to set $v = \frac{dx}{dt}$. Then $\frac{d}{dx}\left(\frac{v^2}{2}\right) = v\frac{dv}{dx} = \frac{dv}{dt} = \frac{d}{dx}\left(\frac{n^2}{2}\right)$. Hence $v^2 = n^2 - a^2$, where a is an arbitrary constant. The last two equations in (2) now imply that $\frac{dy}{dt} = \pm a$. Since $\frac{dx}{dt} = v = \pm(n^2 - a^2)^{1/2}$ it follows that

$$\frac{dy}{dx} = \pm \frac{a}{(n^2 - a^2)^{1/2}}, \tag{3}$$

and

$$y(x) = y(x_0) \pm \int_{x_0}^{x} \frac{a\,dx}{(n^2 - a^2)^{1/2}}. \tag{4}$$

The rays emanating from the source will be called "incident" rays. Let $\tan\alpha$ be the slope of such a ray at the source. Then it follows from (3) that

$$a = n(x_0)\sin\alpha. \tag{5}$$

If $\begin{Bmatrix} 0 \le \alpha < \pi/2 \\ \pi/2 < \alpha \le \pi \end{Bmatrix}$ the incident rays proceed to the $\begin{Bmatrix} \text{right} \\ \text{left} \end{Bmatrix}$ and are given by

$$y = y^{i}(x) = \pm\int_{x_0}^{x} \frac{a\,dx}{(n^2 - a^2)^{1/2}} = \int_{x<}^{x>} \frac{a\,dx}{(n^2 - a^2)^{1/2}}\,. \tag{6}$$

Here $x<$ and $x>$ denote respectively the smaller and larger of x and x_0.

For $\pi/2 < \alpha \le \pi$ some incident rays hit the boundary and are reflected while the others become vertical and are turned back to the right before hitting the boundary. We see from (3) and (5) that such <u>turning points</u> occur when $n(x) = a = n(x_0)\sin\alpha$. Beyond the turning point such rays will be called <u>refracted rays</u>. A particular ray, with $\alpha = \alpha_b$, is tangent to the boundary; i.e., its turning point is at $x = x_b$. Thus α_b is determined by the equation

$$n(x_b) = n(x_0)\sin\alpha_b. \tag{7}$$

This "limiting ray" is illustrated in the figure. Its continuation lies on the shadow boundary. Incident rays with $\pi/2 < \alpha < \alpha_b$ produce refracted rays at their turning points. Incident rays with $\alpha_b < \alpha \le \pi$ produce reflected rays at the boundary. In addition the limiting ray produces a surface ray on the boundary and diffracted rays in the shadow region.

In order to calculate the phase on an incident ray we use (2.12).

Thus we obtain

$$s^1 = \int_{t_o}^{t} n^2[x(t)]dt = \int_{x<}^{x>} \frac{n^2\, dx}{(n^2-a^2)^{\frac{1}{2}}} . \qquad (8)$$

Here we have used the identity, $dx = \pm(n^2 - a^2)^{\frac{1}{2}}dt$, which was derived above equation (3). (6) and (8) now yield

$$s^1 = \int_{x<}^{x>} (n^2 - a^2)^{\frac{1}{2}}\, dx + \int_{x<}^{x>} \frac{a^2 dx}{(n^2 - a^2)^{\frac{1}{2}}} = ay + \int_{x<}^{x>} (n^2 - a^2)^{\frac{1}{2}}\, dx . \quad (9)$$

The wave produced by an isotropic line source was derived in Section 9. Therefore from (9.10) we may conclude that

$$z_o^1 = e^{i\pi/4} \left[\frac{1}{8\pi k n(x)} \frac{d\alpha}{dv(x)} \right]^{\frac{1}{2}} . \qquad (10)$$

In order to calculate $\frac{d\alpha}{dv(x)}$ we see from (5) and (6) that

$$\frac{dy}{d\alpha} = \frac{dy}{da} \frac{da}{d\alpha} = \left[n^2(x_o) - a^2\right]^{\frac{1}{2}} \int_{x<}^{x>} \frac{n^2 dx}{(n^2 - a^2)^{3/2}} , \qquad (11)$$

and from the figure we see that

$$\cos \gamma = (1 + \tan^2\gamma)^{-1/2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-\frac{1}{2}} = n^{-1}(n^2 - a^2)^{\frac{1}{2}} . \qquad (12)$$

Since $dv = dy \cos \gamma$ it follows that

$$n \frac{dv}{d\alpha} = n \frac{dy}{d\alpha} \cos \gamma = \left[n^2 - a^2\right]^{\frac{1}{2}} \left[n^2(x_o) - a^2\right]^{\frac{1}{2}} \int_{x<}^{x>} \frac{n^2 dx}{(n^2 - a^2)^{3/2}} . \quad (13)$$

By inserting (13) in (10) we obtain

$$z_0^i = e^{i\pi/4}\left[n^2(x_0) - \alpha^2\right]^{-1/4}\left[n^2 - \alpha^2\right]^{-1/4}\left[8\pi k\int_{x_<}^{x_>}\frac{n^2\,dx}{(n^2-\alpha^2)^{3/2}}\right]^{-1/2}. \tag{14}$$

Thus the leading term of the incident wave is given by (9), (14) and

$$u^i \sim e^{iks^i}\, z_0^i. \tag{15}$$

For $\alpha_b < \alpha \leq \pi$ the incident ray hits the boundary. The corresponding reflected ray is obtained by reflecting the incident ray across the horizontal line $y = y_1(x_b)$. Therefore it is given by

$$y = y^r(x) = 2y_1(x_b) - y_1(x) = \left\{\int_{x_b}^{x_0} + \int_{x_b}^{x}\right\}\frac{\alpha\,dx}{(n^2-\alpha^2)^{1/2}}. \tag{16}$$

The phase on the reflected ray is obtained by an argument similar to that which led to (9). The result is

$$\begin{aligned} s^r &= s^i(x_b) + \alpha\left[y - y^i(x_b)\right] + \int_{x_b}^{x}(n^2-\alpha^2)^{1/2}\,dx \\ &= \alpha y + \left\{\int_{x_b}^{x_0} + \int_{x_b}^{x}\right\}(n^2-\alpha^2)^{1/2}\,dx. \end{aligned} \tag{17}$$

In order to determine the reflected amplitude, we first use (10.8). This yields

$$z_0^r(x_b) = r z_0^i(x_b); \quad r = \frac{n(x_b)\cos\theta - \alpha}{n(x_b)\cos\theta + \alpha}. \tag{18}$$

Here θ is the angle of incidence (or angle of reflection). As in (12) we see that

$$\cos\theta = n^{-1}(x_b)\,[n^2(x_b)-a^2]^{1/2}, \tag{19}$$

Hence the reflection coefficient r is given by

$$r = \frac{[n^2(x_b) - a^2]^{1/2} - a}{[n^2(x_b) - a^2]^{1/2} + a}. \tag{20}$$

Next we use (3.7) and (3.8) to obtain

$$z_o^r = z_o^r(x_b)\left[\frac{n(x_b)\,dv(x_b)}{n(x)\,dv(x)}\right]^{1/2} \tag{21}$$

and then (21), (18), and (14) yield

$$z_o^r = re^{i\pi/4}\,[n^2(x_o)-a^2]^{-1/4}\,[n^2(x_b)-a^2]^{-1/4} \times\left[8ia\int_{x_b}^{x_o}\frac{n^2\,dx}{(n^2-a^2)^{3/2}}\right]^{-1/2}\left[\frac{n(x_b)\,dv(x_b)}{n(x)\,dv(x)}\right]^{1/2}. \tag{22}$$

As in (13) we can show that

$$n(x)\frac{dv(x)}{dx} = [n^2-a^2]^{1/2}\,[n^2(x_o)-a^2]^{1/2}\left\{\int_{x_b}^{x_o}+\int_{x_b}^{x}\right\}\frac{n^2\,dx}{(n^2-a^2)^{3/2}}, \tag{23}$$

hence

$$\frac{n(x_b)\,dv(x_b)}{n(x)\,dv(x)} = \frac{[n^2(x_b)-a^2]^{1/2}}{[n^2-a^2]^{1/2}}\int_{x_b}^{x_o}\frac{n^2\,dx}{(n^2-a^2)^{3/2}}\left[\left\{\int_{x_b}^{x_o}+\int_{x_b}^{x}\right\}\frac{n^2\,dx}{(n^2-a^2)^{3/2}}\right]^{-1}. \tag{24}$$

By inserting (24) in (22) we obtain

$$z_o^r = re^{i\pi/4}[n^2(x_o) - a^2]^{-1/4}\,[n^2(x) - a^2]^{-1/4}$$

$$\cdot \left[2ik \left\{ \int_{x_b}^{x_o} + \int_{x_b}^{x} \right\} \frac{n^2 dx}{(n^2 - a^2)^{3/2}} \right]^{-\frac{1}{2}} . \qquad (25)$$

The leading term of the reflected wave is given by (17), (20), (25), and

$$u^r \sim e^{iks^r} z_o^r . \qquad (26)$$

In order to determine the refracted wave we consider the incident rays with starting angle in the interval $\pi/2 \leq \alpha < \alpha_b$. The value x_α of x at the turning point is given by

$$n(x_\alpha) = a = n(x_o) \sin \alpha . \qquad (27)$$

Up to the turning point, (6), (9), and (14) are valid, but (14) is indeterminate at $x = x_\alpha$. Therefore we must determine the limit

$$\lambda = \lim_{x \to x_\alpha} \left[n^2(x) - a^2\right]^{1/2} \int_x^{x_0} \frac{n^2(x')dx'}{\left[n^2(x')-a^2\right]^{3/2}} . \tag{28}$$

To do this we set $z = x - x_\alpha$, $z' = x' - x_\alpha$ and introduce the expansion

$$n^2(x) = n^2(x_\alpha) + 2nn'(x_\alpha)(x-x_\alpha) + \ldots = a^2 + bz + \ldots, \; b = 2nn'(x_\alpha). \tag{29}$$

Then

$$\lambda = \lim_{z \to 0} \left[b^{1/2} z^{1/2} + \ldots\right] \left\{ \int_z^{x_0 - x_\alpha} \left[a^2 (bz')^{-3/2} + \ldots\right] dz' \right\} \tag{30}$$

$$= -\frac{2a^2}{b} \lim_{z \to 0} z^{1/2} \left[(z')^{-\frac{1}{2}} + \ldots\right]_z^{x_0 - x_\alpha} = 2\frac{a^2}{b} = \frac{n(x_\alpha)}{n'(x_\alpha)} .$$

Thus if we let $x \to x_\alpha$ in (14) the result is

$$r_0^1(x_\alpha) = e^{i\pi/4} k^{1/2} (8\pi\kappa)^{-1/2} \left[n^2(x_0) - a^2\right]^{-1/4}, \; \kappa = \frac{n'(x_\alpha)}{n(x_\alpha)} . \tag{31}$$

The quantity κ has an interesting geometric interpretation. To see this we write the ray equations (2.6) in the vector form

$$n^2 \ddot{X} + n\dot{n} \dot{X} = \nabla(\tfrac{1}{2}n^2); \quad \dot{} = \frac{d}{ds} . \tag{32}$$

Since s is arclength, $\ddot{X} = \kappa N$, where N is the unit normal vector and κ is the curvature of the ray. Multiplication of (32) by N yields

$$n^2\kappa = N\cdot\nabla(\tfrac{1}{2}n^2) = nN\cdot\nabla n. \qquad (33)$$

We now apply (33) to the incident ray at the turning point x_α. At this point, $N = (1,0)$, hence $N\cdot\nabla n = n'(x_\alpha)$ and $\kappa = n'(x_\alpha)/n(x_\alpha)$. Thus we see that in (31) κ is the curvature of the ray at the turning point.

Beyond the turning point, the refracted ray is given by (16) with x_b replaced by x_α. Similarly the phase is given by (17) with x_b replaced by x_α. The amplitude on the refracted ray can be calculated by our earlier method although a technical difficulty arises in computing $\frac{dy}{d\alpha}$ from the ray formula. (An integration by parts must first be performed since straightforward differentiation leads to an indeterminate form.). The details will be omitted here.

We now consider the diffracted field in the shadow region. The limiting ray is tangent to the boundary at the point $Q_1 = (x_b, y'_b)$. It gives rise to a surface ray which proceeds along the boundary $x = x_b$ as a straight line. At each point $P_1 = (x_b, y_b)$ the surface ray sheds a diffracted ray. For these rays, $a = n(x_o)\sin\alpha_b = n(x_b)$. Hence from (4) the diffracted rays are given by

$$y = y^d(x) = y_b + \int_{x_b}^{x} \frac{n(x_b)dx}{[n^2 - n^2(x_b)]^{1/2}} . \qquad (34)$$

Thus they form a one-parameter family of congruent curves.

In order to apply the formula (18.14) for the diffracted field we must evaluate the limit (18.15). From (34) we see that $dy = dy_b$, hence

$dw = dy \cos\gamma = dy_b \cos\gamma$. Let $P' = (x', y')$ be a point on the diffracted ray joining P_1 and $P = (x,y)$. Then

$$\frac{da(P')}{da(P)} = \frac{dw(P')}{dw(P)} = \frac{\cos\gamma(P')}{\cos\gamma(P)} . \tag{35}$$

In (18.15) σ_0 denotes the distance from P_1 to P' along the ray. Since $\gamma = \pi/2$ when $\sigma_0 = 0$ we see that

$$\lim_{P' \to P_1} \frac{\cos\gamma(P')}{\sigma_0} = \lim_{\sigma_0 \to 0} \frac{\cos\gamma(\sigma_0) - \cos\gamma(0)}{\sigma_0} = \frac{d\cos\gamma}{d\sigma_0}\bigg|_{\sigma_0 = 0}$$
$$= -\frac{d\gamma}{d\sigma_0} = \kappa. \tag{36}$$

Here κ is the curvature of the diffracted ray (hence the curvature of the limiting ray) at the boundary. Now from (12), $n(P)\cos\gamma(P) = (n^2 - a^2)^{1/2}$ $= \left[n^2(x) - n^2(x_b)\right]^{1/2}$, therefore from (18.15), (35), and (36)

$$\frac{d\bar{a}(P_1)}{n(P)da(P)} = \lim \frac{\cos\gamma(P')}{n(P)\sigma_0 \cos\gamma(P)} = \frac{\kappa}{n(P)\cos\gamma(P)} = \kappa\left[n^2(x) - n^2(x_b)\right]^{-\frac{1}{2}} . \tag{37}$$

(18.14) now yields

$$u^d(P) = z_0^1(Q_1) \exp\left\{ik\left[s^1(Q_1) + \int_{Q_1}^{P_1} n\,d\sigma + \int_{P_1}^{P} n\,d\sigma\right]\right\}\left[\frac{dw(Q_1)}{dw(P_1)}\, n(Q_1)\,\kappa\right]^{1/2}$$
$$\times\left[n^2(x) - n^2(x_b)\right]^{-1/4}\sum_j k^{-\frac{1}{6}} d_j(P_1) d_j(Q_1) \exp\left\{-\int_{Q_1}^{P_1} \alpha_j(\sigma)\,d\sigma\right\}. \tag{38}$$

Since the surface rays are straight lines $\frac{dw(Q_1)}{dw(P_1)} = 1$. Furthermore $n(Q_1) = n(x_b)$ and $\int_{Q_1}^{P_1} n d\sigma = n(x_b)(y_b - y_b')$. The surface S is a plane on which n and z are constant. Therefore the diffraction coefficients at P_1 and Q_1 are equal and the decay exponent α_j is constant. Thus

$$\int_{Q_1}^{P_1} \alpha_j(\sigma)d\sigma = (y_b - y'_b)\alpha_j.$$

To calculate the change in phase along the diffracted ray we use (34) to obtain

$$d\sigma = [dy^2 + dx^2]^{1/2} = \frac{n\, dx}{[n^2 - n^2(x_b)]^{1/2}}, \tag{39}$$

and

$$\int_{P_1}^{P} n d\sigma = \int_{x_b}^{x} \frac{n^2 dx}{[n^2 - n^2(x_b)]^{1/2}} = \int_{x_b}^{x} [n^2 - n^2(x_b)]^{1/2} dx + \int_{x_b}^{x} \frac{n^2(x_b) dx}{[n^2 - n^2(x_b)]^{1/2}}$$
$$= \int_{x_b}^{x} [n^2 - n^2(x_b)]^{1/2} dx + n(x_b)(y - y_b). \tag{40}$$

To obtain $z_0^1(Q_1)$ we must use (31) rather than (14) which is indeterminate at Q_1. Thus

$$z_0^1(Q_1) = e^{i\pi/4} \kappa^{1/2} (8\pi k)^{-1/2} [n^2(x_0) - n^2(x_b)]^{-1/4}. \tag{41}$$

From (9) (42)

$$s^1(Q_1) = n(x_b)y_b' + \int_{x_b}^{x_o} [n^2 - n^2(x_b)]^{1/2}\, dx.$$

We may now insert all these results in (38). This yields

(43)

$$u^d(P) \sim [n^2(x_o) - n^2(x_b)]^{-1/4}\, [n^2(x) - n^2(x_b)]^{-1/4} \left[\frac{n(x_b)}{8\pi}\right]^{1/2} \frac{\kappa}{k}$$

$$\times \exp\left\{ ik\left[n(x_b)y + \left\{\int_{x_b}^{x_o} + \int_{x_b}^{x}\right\}[n^2 - n^2(x_b)]^{1/2}\, dx\right] + i\pi/4 \right\}$$

$$\times \sum_j d_j^2 \exp\left\{ -(y_b - y_b')\alpha_j \right\}.$$

The diffraction coefficient d_j and the decay exponent α_j are obtained from (19.13 - 15) by setting $a^{-1}(X) = 0$, $\kappa(X) = \kappa = n'(x_b)/n(x_b)$, and $n(X) = n(x_b)$. Thus

$$\alpha_j = e^{-i\pi/6}\; q_j\, [6^{-1} k n(x_b)]^{1/3}\; \kappa^{2/3}, \tag{44}$$

and

$$d_j = -e^{i\pi/24}\; 6^{-1/6}\, 2^{-1/4}\, \pi^{3/4}[k n(x_b)]^{1/6}\; \kappa^{-1/6} \tag{45}$$

$$\times\left[q_j A^2(q_j) + 3\left\{A'(q_j)\right\}^2\right]^{-1/2}.$$

Here

$$\frac{A'(q_j)}{A(q_j)} = e^{5\pi i/6}\; ak^{1/3}\left\{6n^2(x_b)\kappa\right\}^{-1/3}. \tag{46}$$

The problem discussed in this section can be solved exactly by separation of variables. The asymptotic expansion of the solution can be obtained by applying asymptotic methods (the "W.K.B. method") for ordering differential equations. When this is done the results agree with those we have derived (see [37]).

B. Asymptotic Methods for Maxwell's Equations

B1. Time-harmonic Solutions of Maxwell's equations

Although solutions of the wave equation are frequently used to describe optical phenomena, it is well known that a rigorous description of optical and other electromagnetic phenomena can be obtained only by solving Maxwell's equations for the electromagnetic field. At high frequencies, asymptotic methods are particularly useful for this purpose. We shall see that many features of the asymptotic method for solving Maxwell's equations are similar to those which we have examined for the reduced wave equation, and we shall make full use of the similarity. Nevertheless the vector character of the electromagnetic problem introduces significant differences which we shall examine in detail. For the material in this chapter we are largely indebted to R.K. Luneburg [34].

Proceeding as in chapter A we shall assume harmonic time dependence and derive the time reduced form of Maxwell's equations. Then an asymptotic series will be inserted into these equations, and equations for phase and amplitude functions will be derived. We will see that the phase function again satisfies the eiconal equation. For this reason much of our earlier work will be directly applicable. In particular we shall have the same ray equations. The main difference in the electromagnetic theory lies in the vector character of the amplitude. However, even here the essential feature remains: The components of the amplitude functions satisfy ordinary differential equations (transport equations) along the rays.

In M.K.S. units [38], Maxwell's equations take the form*

$$\nabla \times \hat{\mathcal{H}} - \frac{\partial}{\partial t}(\epsilon \hat{\mathcal{E}}) = \sigma_1 \hat{\mathcal{E}}, \tag{1}$$

$$\nabla \times \hat{\mathcal{E}} + \frac{\partial}{\partial t}(\mu \hat{\mathcal{H}}) = 0, \tag{2}$$

$$\nabla \cdot (\mu \hat{\mathcal{H}}) = 0, \tag{3}$$

$$\nabla \cdot (\epsilon \hat{\mathcal{E}}) = \rho \,. \tag{4}$$

Here $\hat{\mathcal{E}}(X,t)$, $\hat{\mathcal{H}}(X,t)$, are the (real) electric and magnetic field vectors, and $\epsilon(X)$, $\mu(X)$, and $\sigma_1(X)$ are the dielectric "constant", magnetic permeability, and conductivity of the medium. $\rho(X,t)$ is the electric charge density. ϵ, μ, and σ_1 are assumed to be piece-wise smooth functions of X.

We shall be interested in time-harmonic fields, of the form

$$\hat{\mathcal{E}}(X,t) = \mathrm{Re}\left[\mathcal{E}(X)e^{-i\omega t}\right], \hat{\mathcal{H}}(X,t) = \mathrm{Re}\left[\mathcal{H}(X)e^{-i\omega t}\right] . \tag{5}$$

Then it is easy to see that (1) and (2) are satisfied, provided the (complex) vectors $\mathcal{E}, \mathcal{H}$ satisfy the time-reduced equations

$$\nabla \times \mathcal{H} + i\omega\epsilon\, \mathcal{E} = \sigma_1 \mathcal{E}\,; \quad \nabla \times \mathcal{E} - i\omega\mu \mathcal{H} = 0. \tag{6}$$

From the second equation it follows immediately that $\nabla \cdot (\mu \mathcal{H}) = 0$, so (3) is automatically satisfied. (4) may be thought of simply as a definition of ρ.

*In our notation we have reserved the symbols E and H for the leading term of the amplitudes of the electric and magnetic field vectors. This accounts for the unorthodox notation in the first few equations.

Let

$$\mathcal{E} = A + iB, \quad \bar{\mathcal{E}} = A - iB. \tag{7}$$

Here A, B are vectors with real components and the bar denotes the complex conjugate. Thus from (5)

$$\hat{\mathcal{E}} = \frac{1}{2}\left(\mathcal{E} e^{-i\omega t} + \bar{\mathcal{E}} e^{i\omega t}\right) = A \cos \omega t + B \sin \omega t. \tag{8}$$

It follows from this equation that as t varies (at each point X) the vector $\hat{\mathcal{E}}(X,t) = (\hat{\mathcal{E}}_1, \hat{\mathcal{E}}_2, \hat{\mathcal{E}}_3)$ describes an ellipse which lies in the plane determined by A and B. This <u>plane of polarization</u> is therefore perpendicular to the vector

$$B \times A = \frac{1}{2i}\left[iB \times A - iA \times B\right] = \frac{1}{2i}\,\mathcal{E} \times \bar{\mathcal{E}}. \tag{9}$$

The principal axes of the ellipse correspond to the extreme values of

$$\hat{\mathcal{E}}^2 = \frac{1}{4}\left(\mathcal{E}^2 e^{-2i\omega t} + \bar{\mathcal{E}}^2 e^{2i\omega t} + 2\,\mathcal{E} \cdot \bar{\mathcal{E}}\right). \tag{10}$$

Equating to zero the t derivative of (10) yields

$$e^{2i\omega t} = \pm\sqrt{\mathcal{E}^2/\bar{\mathcal{E}}^2}. \tag{11}$$

If we insert (11) in (10) we see that the extreme values of $\hat{\mathcal{E}}^2$ are

$$\hat{\mathcal{E}}^2 = \frac{1}{2}\left(\mathcal{E} \cdot \bar{\mathcal{E}} \pm \sqrt{\mathcal{E}^2 \bar{\mathcal{E}}^2}\right). \tag{12}$$

The ratio

$$\delta^2 = \frac{\mathcal{E}\cdot\bar{\mathcal{E}} - \sqrt{\mathcal{E}^2\bar{\mathcal{E}}^2}}{\mathcal{E}\cdot\bar{\mathcal{E}} + \sqrt{\mathcal{E}^2\bar{\mathcal{E}}^2}} \tag{13}$$

is called the ellipticity. The polarization is circular if $\delta = 1$, i.e., if

$$\mathcal{E}^2 = \bar{\mathcal{E}}^2 = 0, \tag{14}$$

and linear if $\delta = 0$, i.e., if

$$\begin{aligned} 0 &= (\mathcal{E}\cdot\bar{\mathcal{E}})^2 - \mathcal{E}^2\bar{\mathcal{E}}^2 = (\mathcal{E}\times\bar{\mathcal{E}})\cdot(\bar{\mathcal{E}}\times\mathcal{E}) \\ &= (\mathcal{E}\times\bar{\mathcal{E}})\cdot\overline{(\mathcal{E}\times\bar{\mathcal{E}})}. \end{aligned} \tag{15}$$

However, for any complex vector, $C = (C_1, C_2, C_3)$, $C\cdot\bar{C} = |C_1|^2 + |C_2|^2 + |C_3|^2$ can vanish only if $C = 0$. Hence the polarization is linear if and only if

$$\mathcal{E}\times\bar{\mathcal{E}} = 0. \tag{16}$$

It is easy to show that (16) is equivalent to the condition

$$\mathcal{E} = aG \tag{17}$$

where a is a (complex) scalar and G is a real vector.

The electromagnetic energy density is defined by

$$\hat{v} = \frac{1}{2}\left(\epsilon \hat{\mathcal{E}}^2 + \mu \hat{\mathcal{H}}^2\right) \tag{18}$$

and the energy flux vector (Poynting vector) is defined by

$$\hat{S} = \hat{\mathcal{E}} \times \hat{\mathcal{H}}. \tag{19}$$

We define the corresponding time-averaged quantities

$$\tilde{v} = \frac{1}{2\tau} \int_0^{\tau} \left(\epsilon \hat{\mathcal{E}}^2 + \mu \hat{\mathcal{H}}^2\right) dt, \tag{20}$$

$$\tilde{S} = \frac{1}{\tau} \int_0^{\tau} \left(\hat{\mathcal{E}} \times \hat{\mathcal{H}}\right) dt. \tag{21}$$

From (8) and the analogous equation for $\hat{\mathcal{H}}$ it is easy to show that

$$\tilde{v} = \frac{1}{4}\left[\epsilon\, \mathcal{E} \cdot \bar{\mathcal{E}} + \mu\, \mathcal{H} \cdot \bar{\mathcal{H}}\right], \tag{22}$$

$$\tilde{S} = \frac{1}{4}\left[\mathcal{E} \times \bar{\mathcal{H}} + \bar{\mathcal{E}} \times \mathcal{H}\right]. \tag{23}$$

These equations hold provided $\tau = \frac{2\pi j}{\omega}$, where j is a positive integer, or $\tau \to \infty$.

B2. <u>Asymptotic solution of the reduced equations</u>

In empty space $\epsilon(X)$ and $\mu(X)$ have the constant values $\epsilon_0 = 8.854 \times 10^{-12}$ farad/meter and $\mu_0 = 1.257 \times 10^{-6}$ henry/meter [38]. The constant $c_0 = (\epsilon_0 \mu_0)^{-1/2} = 2.99790 \times 10^{8}$ meter/sec is the familiar "speed of light". As in chapter A we introduce the propagation constant (or wave number) $k = \omega/c_0$, and we assume that the complex vectors $\mathcal{E}$, $\mathcal{H}$ have asymptotic expansions of the form

$$\mathcal{E} \sim e^{iks} \sum_{m=0}^{\infty} (ik)^{-m} E_m, \quad \mathcal{H} \sim e^{iks} \sum_{m=0}^{\infty} (ik)^{-m} H_m. \tag{1}$$

The real scalar function $s(X)$ is again called the phase function (or phase). If we insert (1) in (1.6) and collect coefficients of the same powers of (ik) we obtain

$$\nabla s \times H_m + \nabla \times H_{m-1} + c_0 \epsilon E_m = \sigma_1 E_{m-1}, \quad \nabla s \times E_m + \nabla \times E_{m-1} - c_0 \mu H_m = 0;$$

(?)

$m = 1, 2, \ldots$.

The equations for $m = 0$ are

$$\nabla s \times H + c_0 \epsilon E = 0, \quad \nabla s \times E - c_0 \mu H = 0. \tag{3}$$

Here, and in all subsequent equations, we omit the subscript zero.

We see at once from (3) that

$$E \cdot H = E \cdot \nabla s = H \cdot \nabla s = 0 \quad . \tag{4}$$

By eliminating H from (3) we obtain

$$c_0^2 \epsilon\mu E = - \nabla s \times (\nabla s \times E) = (\nabla s)^2 E. \tag{5}$$

It follows that, if E is non-zero, $s(x)$ must satisfy the eiconal equation

$$(\nabla s)^2 = n^2(X). \tag{6}$$

Here $n(X)$ is the index of refraction of the medium, defined by the equation

$$n^2 = c_o^2\,\varepsilon\mu = \frac{\varepsilon\mu}{\varepsilon_o\mu_o} = \frac{c_o^{\,2}}{c^2(X)}\,; \quad c^2(X) = \frac{1}{\varepsilon(X)\mu(X)}\,. \tag{7}$$

We note the important fact that the phase $s(X)$ again satisfies the eiconal equation. It follows that the main features (rays, wave-fronts, etc.) of our expansion will be the same as those of chapter A. In particular the results of sections A2 and A6 can be carried over unchanged.

B3. The transport equations for the amplitude

If we insert (2.1) into (1.22) and (1.23) we see that $\tilde{v} = v + O(k^{-1})$ and $\tilde{S} = S + O(k^{-1})$ where

$$v = \frac{1}{4}\left[\varepsilon E\cdot\bar{E} + \mu H\cdot\bar{H}\right], \tag{1}$$

$$S = \frac{1}{4}\left[E\times\bar{H} + \bar{E}\times H\right]. \tag{2}$$

Now from (2.3),

$$c_o\varepsilon E\cdot\bar{E} = (H\times\nabla s)\cdot\bar{E}, \quad c_o\mu H\cdot\bar{H} = (\bar{E}\times\nabla s)\cdot H, \tag{3}$$

$$E\times\bar{H} = \bar{E}\times H = \frac{1}{c_o\varepsilon}\,(H\cdot\bar{H})\nabla s. \tag{4}$$

In (3) the left sides are real. Thus the right sides, which are conjugates of each other, must be equal. It follows that

$$v = \frac{1}{2}\,\epsilon\, E\cdot\bar{E} = \frac{1}{2}\,\mu\, H\cdot\bar{H}, \tag{5}$$

$$S = \frac{1}{2\epsilon c_0}(H\cdot\bar{H})\nabla s = \frac{v}{\epsilon\mu c_0}\nabla s = \frac{c_0 v}{n^2}\nabla s, \tag{6}$$

and

$$c_0 v = S\cdot\nabla s. \tag{7}$$

In order to obtain differential equations for E and H along a ray, we return to (2.2) for $m = 1$. For convenience, we set $\nabla s = R$ and symmetrize the equations by introducing a fictitious magnetic conductivity σ_2 (which later will be set equal to zero). Then (2.2) yields

$$R\times H_1 + c_0\epsilon E_1 = -\nabla\times H + \sigma_1 E, \quad R\times E_1 - c_0\mu H_1 = -\nabla\times E - \sigma_2 H. \tag{8}$$

These equations are symmetric under replacement of

$$E, H, \epsilon, \mu, \sigma_1, \sigma_2 \quad \text{by } H, E, -\mu, -\epsilon, -\sigma_2, -\sigma_1. \tag{9}$$

They also imply new conditions on E and H. To obtain these conditions, we first note that $R\times(R\times E_1) = (R\cdot E_1)R - c_0^2\epsilon\mu E_1$, since $R^2 = c_0^2\epsilon\mu$. If we multiply the first equation (8) by $c_0\mu$ and add it to the vector product of R with the second equation we obtain

$$(R\cdot E_1)R = -c_0\mu\nabla\times H - R\times(\nabla\times E) - \sigma_2 R\times H + c_0\mu\,\sigma_1 E. \tag{10}$$

But from (2.3)

$$R \times H + c_0 \epsilon E = 0, \quad R \times E - c_0 \mu H = 0. \tag{11}$$

Hence

$$R \times \left\{ \nabla \times \left(\frac{1}{\mu} R \times E\right) + \frac{1}{\mu} R \times (\nabla \times E) - c_0\left(\frac{\sigma_2 \epsilon}{\mu} + \sigma_1\right) E \right\} = 0. \tag{12}$$

Thus we have obtained an equation involving E from which E_1, H_1, and H have been eliminated.

Now $\nabla \times R = \nabla \times (\nabla s) = 0$ so $\nabla \times (\frac{1}{\mu} R) = \nabla(\frac{1}{\mu}) \times R$. Hence

$$R \times \left\{ E \times \left(\nabla \times \frac{1}{\mu} R\right)\right\} = R \times \left\{ E \times \left[\nabla\left(\frac{1}{\mu}\right) \times R\right] \right\} = R \times \left\{ -\left[E \cdot \nabla\left(\frac{1}{\mu}\right)\right] R \right\} = 0 \tag{13}$$

Thus (12) may be written as

$$R \times \left\{ \nabla \times \left(\frac{1}{\mu} R \times E\right) + \frac{1}{\mu} R \times (\nabla \times E) + E \times \left(\nabla \times \frac{1}{\mu} R\right) - c_0\left(\frac{\sigma_2 \epsilon}{\mu} + \sigma_1\right) E \right\} = 0. \tag{14}$$

We now set $\frac{1}{\mu} R = A$ and use the vector identity

$$\nabla \times (A \times E) + A \times (\nabla \times E) + E \times (\nabla \times A) = -2\left(A_1 \frac{\partial E}{\partial x_1} + A_2 \frac{\partial E}{\partial x_2} + A_3 \frac{\partial E}{\partial x_3}\right) - E \nabla \cdot A + A \nabla \cdot E + \nabla(A \cdot E). \tag{15}$$

Since $R \cdot E = 0$ it follows that (14) is equivalent to

$$R \times \left\{ 2\left(R_1 \frac{\partial E}{\partial x_1} + R_2 \frac{\partial E}{\partial x_2} + R_3 \frac{\partial E}{\partial x_3}\right) + \mu E \nabla\cdot\left(\frac{1}{\mu} R\right) + c_0(\epsilon\, \sigma_2 + \mu\, \sigma_1)E \right\} = 0. \tag{16}$$

Let $X = X(\tau)$ be the equation of a ray, and let us choose the parameter τ so that $|\dot{X}| = \left|\frac{dx}{d\tau}\right| = n$. [See the ray equations (A2.8), (A2.9)]. Then

$$\dot{X} = \nabla u = R. \tag{17}$$

Thus $(R\cdot\nabla)E = (\dot{X}\cdot\nabla)E = \frac{dE}{d\tau}$. We use this in (16) and note that the quantity in braces must be parallel to R. Therefore (16) may be written as

$$\frac{dE}{d\tau} + \frac{\mu}{2} E \nabla\cdot\left(\frac{1}{\mu} R\right) + \frac{c_0}{2}(\epsilon\, \sigma_2 + \mu\sigma_1)E = \beta R \;. \tag{18}$$

Here β is to be determined. However, $R\cdot R = n^2$ and $R\cdot E = 0$. Therefore, by scalar multiplication of (18) by R, we obtain

$$\beta = \frac{1}{n^2}\left(R\cdot\frac{dE}{d\tau}\right) = -\frac{1}{n^2}\left(E\cdot\frac{dR}{d\tau}\right). \tag{19}$$

Furthermore, from (A2.8)

$$\frac{dR}{d\tau} = \ddot{X} = \frac{1}{2}\nabla(n^2) = n\nabla n. \tag{20}$$

Now (19) and (20) yield

$$\beta = -\frac{E\cdot\nabla n}{n}. \tag{21}$$

We next introduce the notation

$$\mu\nabla\cdot\left(\frac{1}{\mu}R\right) = \mu\nabla\cdot\left(\frac{1}{\mu}\nabla s\right) = \mu\sum_j\left(\frac{1}{\mu}s_{x_j}\right)_{x_j} = \Delta_\mu s. \tag{22}$$

(If μ = const, Δ_μ is the Laplacian operator). Now (18) becomes

$$\frac{dE}{d\tau} + \frac{1}{2}E\,\Delta_\mu s + \left(\frac{E\cdot\nabla n}{n}\right)\nabla s + \frac{c_o}{2}(\epsilon\sigma_2 + \mu\sigma_1)E = 0. \tag{23}$$

By means of the symmetry property (9) we obtain an equation analogous to (23) for H. We then set $\sigma_2 = 0$ in (23) and in this equation to obtain

$$\frac{dE}{d\tau} + \frac{1}{2}(\Delta_\mu s + c_o\sigma_1\mu)E + \left(\frac{E\cdot\nabla n}{n}\right)\nabla s = 0, \tag{24}$$

$$\frac{dH}{d\tau} + \frac{1}{2}(\Delta_\epsilon s + c_o\sigma_1\mu)H + \left(\frac{H\cdot\nabla n}{n}\right)\nabla s = 0. \tag{25}$$

These ordinary differential equations for E and H along a ray can be simplified. To do so we introduce the vector

$$F = \frac{1}{\epsilon\mu}R = \frac{c_o^2}{n^2}\nabla s, \tag{26}$$

Then, since $\frac{d}{d\tau}\log\epsilon = \frac{1}{\epsilon}\nabla\epsilon\cdot\dot{X} = \frac{1}{\epsilon}\nabla\epsilon\cdot R = \mu F\cdot\nabla\epsilon,$

$$\Delta_\mu s = \mu\nabla\cdot(\epsilon F) = \frac{n^2}{c_o^2}\nabla\cdot F + \mu F\cdot\nabla\epsilon = \frac{n^2}{c_o^2}\nabla\cdot F + \frac{d}{d\tau}\log\epsilon\ . \tag{27}$$

By inserting (27) into (24) we obtain

$$\frac{dE}{d\tau} + \frac{1}{2}\left(\frac{d}{d\tau}\log\epsilon + \frac{n^2}{c_o^2}\nabla\cdot F + c_o\sigma_1\mu\right)E + \left(\frac{E\cdot\nabla n}{n}\right)\nabla s = 0. \quad (28)$$

But

$$\frac{d}{d\tau}\left(\sqrt{\epsilon}\,E\right) = \sqrt{\epsilon}\,\frac{dE}{d\tau} + \frac{1}{2\sqrt{\epsilon}}\,\frac{d\epsilon}{d\tau}\,E = \sqrt{\epsilon}\left(\frac{dE}{d\tau} + \frac{1}{2}\frac{d}{d\tau}\log\epsilon\right). \quad (29)$$

Thus (28) becomes

$$\frac{d}{d\tau}\left(\sqrt{\epsilon}\,E\right) + \frac{1}{2}\left(\frac{n^2}{c_o^2}\nabla\cdot F + c_o c_1\mu\right)\sqrt{\epsilon}\,E + \left(\frac{\sqrt{\epsilon}\,E\cdot\nabla n}{n}\right)\nabla s = 0. \quad (30)$$

The analogous equation for H is

$$\frac{d}{d\tau}\left(\sqrt{\mu}\,H\right) + \frac{1}{2}\left(\frac{n^2}{c_o^2}\nabla\cdot F + c_o\sigma_1\mu\right)\sqrt{\mu}\,H + \left(\frac{\sqrt{\mu}\,H\cdot\nabla n}{n}\right)\nabla s = 0. \quad (31)$$

These equations, which determine how E and H vary along a ray, can be replaced by simpler equations for the magnitude and direction of these vectors. To this end we introduce a real scalar function v_1 defined by the differential equation

$$\frac{d}{d\tau}\log v_1 = -\left(\frac{n^2}{c_o^2}\nabla\cdot F + c_o\sigma_1\mu\right) \quad (32)$$

and the initial condition

$$2w_1 = 2w = \epsilon E\cdot\bar{E} = \mu H\cdot\bar{H}, \text{ at } \tau = \tau_0 . \tag{33}$$

Here τ_0 is some point on the ray. Let P, Q be complex vectors defined by the equations

$$\sqrt{\epsilon}\, E = \sqrt{2w_1}\; P, \quad \sqrt{\mu}\, H = \sqrt{2w_1}\; Q . \tag{34}$$

Then P and Q are unitary vectors (i.e., $P\cdot\bar{P} = Q\cdot\bar{Q} = 1$) for $\tau = \tau_0$. When (34) is inserted into it, (30) becomes

$$\frac{1}{2} P \left(\frac{d \log w_1}{d\tau} + \frac{n^2}{c_0^2} \nabla\cdot F + c_0 \sigma_1 \mu \right) + \frac{dP}{d\tau} + \frac{P\cdot\nabla n}{n} \nabla s = 0 \tag{35}$$

and (31) leads to a similar equation for Q. This equation and (35) simplify when (32) is used, and the results are

$$\frac{dP}{d\tau} + \frac{P\cdot\nabla n}{n} \nabla s = 0, \quad \frac{dQ}{d\tau} + \frac{Q\cdot\nabla n}{n} \nabla s = 0; \quad \frac{dP}{d\sigma} + \frac{P\cdot\nabla n}{n^2} \nabla s = 0, \quad \frac{dQ}{d\sigma} + \frac{Q\cdot\nabla n}{n^2} \nabla s = 0. \tag{36}$$

From (34) and (2,4) we see that

$$P\cdot Q = P\cdot\nabla s = Q\cdot\nabla s = 0. \tag{37}$$

If we multiply the first equation in (36) by $\bar{P}$, and use the fact that $\bar{P}\cdot\nabla s = 0$, we obtain $\bar{P}\cdot\frac{dP}{d\tau} = 0$, i.e., $\bar{P}\cdot P = \text{const.}$ In the same way we prove that $\bar{Q}\cdot Q = \text{const.}$, and it follows that P and Q are unitary vectors for all τ. It also follows from (34) that $\epsilon E\cdot\bar{E} = 2w_1\, P\cdot\bar{P} = 2w_1$. Then from (5) we see that

$v_1 = v$ and we may henceforth omit the subscript 1. The differential equations (36) will be further analyzed in section 6. Now we shall examine (32) which determines the zero-order average energy density v.

Since $\frac{d}{d\tau} \log v = \frac{1}{v} \nabla v \cdot \nabla s$, (32) yields

$$\nabla v \cdot \nabla s + v\left[\Delta s + n^2 \nabla\left(\frac{1}{n^2}\right) \cdot \nabla s + c_0\, \sigma_1 \mu\right] = 0 \qquad (38)$$

or

$$\nabla \cdot \left(\frac{v}{n^2} \nabla s\right) + \frac{v}{n^2}\, c_0 \sigma_1 \mu = 0. \qquad (39)$$

We now set

$$q = \exp\left\{c_0 \int_{\tau_0}^{\tau} \sigma_1\, \mu\, d\, \tau'\right\} \qquad (40)$$

and note that

$$\nabla s \cdot \nabla q = \frac{dq}{d\tau} = c_0 \sigma_1 \mu\, q. \qquad (41)$$

Then (39) and (41) yield

$$\nabla \cdot \left(q \frac{v}{n^2} \nabla s\right) = q\left[\nabla \cdot \left(\frac{v}{n^2} \nabla s\right) + \frac{v}{n^2}\, c_0 \sigma_1 \mu\right] = 0. \qquad (42)$$

But (42) is of the same form as equation (A3.4) and hence a simple application of Gauss' theorem, as in section A3, yields

$$\frac{q v \xi^2}{n} = \text{const.} \qquad (43)$$

or

$$\frac{v(\tau)\xi^2(\tau)}{n(\tau)} = \frac{v(\tau_0)\xi^2(\tau_0)}{n(\tau_0)} \exp\left\{-c_0 \int_{\tau_0}^{\tau} \sigma_1 \mu\, d\tau'\right\}. \qquad (44)$$

If σ is an arclength parameter on the ray, (A6.10) shows that $d\sigma = n d\tau$. Hence

$c_o \mu d\tau = \frac{c_o \mu d\sigma}{n} = \sqrt{\frac{\mu}{\epsilon}}\; d\sigma$, and (44) becomes

$$\frac{v(\sigma)\xi(\sigma)}{n(\sigma)} = \frac{v(\sigma_o)\xi(\sigma_o)}{n(\sigma_o)} \exp\left\{-\int_{\sigma_o}^{\sigma} \sigma_1 \sqrt{\frac{\mu}{\epsilon}}\, d\sigma'\right\}. \qquad (45)$$

Equation (45) determines the variation of the zero-order average energy density v along a ray. It is the analogue of the solution (A3.8) of the zero-order transport equation for the reduced wave equation. $\xi(\sigma) = \frac{da(\sigma)}{da(\sigma_1)}$ is the expansion ratio introduced in section A3. The higher order transport equations for Maxwell's equations are analyzed in [27].

Since $v = v_1$ the values of the zero order field vectors, E and H can be determined from (34) and (45) once the polarization vectors, P and Q are found. The equations for P and Q are studied in the next section.

In a medium for which $\sigma_1 = 0$, (45) becomes

$$\frac{v\xi}{n} = \text{constant}. \qquad (46)$$

This equation expresses the well-known principle of energy conservation in a tube of rays. (45) describes the dissipation of energy due to the conductivity of the medium.

B 4. The transport equations for the polarization vectors

According to (1.9) the plane of polarization is perpendicular to the vector $\frac{1}{n} \mathcal{E} \times \dot{\mathcal{E}}$, so to zero order it is perpendicular to the vectors

$\frac{1}{2i} E \times \bar{E}$ and $\frac{1}{2i} P \times \bar{P}$. But

$$\nabla s \times (P \times \bar{P}) = (\nabla s \cdot \bar{P})P - (\nabla s \cdot P)\bar{P} = 0. \tag{1}$$

Therefore, the plane of polarization is perpendicular to ∇s, i.e., perpendicular to the ray. From (1.13) we see that to zero order the ellipticity is given by

$$\delta^2 = \frac{E \cdot \bar{E} - \sqrt{E^2 \bar{E}^2}}{E \cdot \bar{E} + \sqrt{E^2 \bar{E}^2}} = \frac{1 - \sqrt{P^2 \bar{P}^2}}{1 + \sqrt{P^2 \bar{P}^2}}. \tag{2}$$

Equation (3.36) implies that P^2 and $\bar{P}^2$ are constant on a ray. Hence the ellipticity is constant along a ray.

From (1.17) we see that for the case of linear polarization P is proportional to a real vector, i.e.,

$$P = aP_0, \tag{3}$$

where P_0 is real, and a is a complex number of modulus one which is constant on a ray. Furthermore

$$1 = P \cdot \bar{P} = P_0^2, \tag{4}$$

i.e., P_0 is a real unit vector.

From (2.3) and (3.34) we now see that

$$Q = T \times P = a\,Q_0, \qquad Q_0 = T \times P_0, \tag{5}$$

where

$$T = \frac{\nabla s}{|\nabla s|} = \frac{1}{n}\nabla s. \tag{6}$$

T, P_0, and Q_0 are orthogonal unit vectors.

Futhermore, it is easy to see that to zero order

$$\hat{\mathcal{E}} \sim \sqrt{\frac{2\nu}{\varepsilon}}\, \cos[ks-\nu t]P_0, \tag{7}$$

$$\hat{\mathcal{H}} \sim \sqrt{\frac{2\nu}{\mu}}\, \cos[ks-\nu t]Q_0. \tag{8}$$

Here we have absorbed the (constant) phase of a into s which is undetermined up to an additive constant on a ray.

For the case of linear polarization we may replace P by P_0 in (3.36), and write that equation in the form

$$nP_0' + [P_0 \cdot (nX')']X' = 0. \tag{9}$$

Here we have used (A2.5) in the form

$$(nX')' = \nabla n. \tag{10}$$

The prime denotes differentiation with respect to the arclength, σ. However, $P_0 \cdot X' = 0$. Hence $P_0 \cdot (nX')' = P_0 \cdot (nX'' + n'X') = nP_0 \cdot X''$, and (9) becomes

$$P_0' + (P_0 \cdot X'')X' = 0. \tag{11}$$

We next apply the theory of space curves to the ray, and introduce the tangent vector $T = X'$, the principal normal vector $N = \frac{X''}{|X''|}$, and the binormal vector $B = T \times N$. These vectors satisfy the Frenet equations

$$T' = \kappa N, \tag{12}$$

$$N' = -\kappa T + \gamma B, \tag{13}$$

$$B' = -\gamma N. \tag{14}$$

Here κ is the principal curvature and γ is the torsion of the curve. With these formulas, (11) becomes

$$P_0' + \kappa(P_0 \cdot N)T = 0. \tag{15}$$

Since P_0 is normal to T,

$$P_0 = \alpha N + \beta B; \quad \alpha^2 + \beta^2 = 1. \tag{16}$$

If we insert (16) into (15) we obtain

$$\alpha' N + \alpha N' + \beta' B + \beta B' + \kappa\alpha T = 0, \tag{17}$$

or

$$(\alpha' - \gamma\beta)N + (\beta' + \alpha\gamma)B = 0. \tag{18}$$

It follows that

$$\alpha' - \gamma\beta = 0; \ \beta' + \alpha\gamma = 0, \tag{19}$$

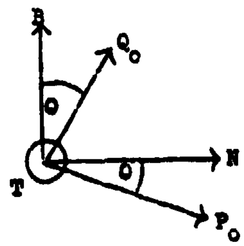
or

$$\frac{d}{d\sigma}(\alpha + i\beta) + i\gamma(\alpha + i\beta) = 0. \tag{20}$$

This equation has the solution

$$\alpha + i\beta = (\alpha_0 + i\beta_0)\, e^{-i\int_{\sigma_0}^{\sigma} \gamma d\sigma'} . \tag{21}$$

Let Θ be the angle between P_0 and N (see figure),



i.e., $\alpha = \cos\Theta$, $\beta = -\sin\Theta$, $\alpha_0 = \cos\Theta_0$, $\beta_0 = -\sin\Theta_0$.

Then (21) becomes

$$\Theta = \Theta_0 + \int_{\sigma_0}^{\sigma} \gamma d\sigma' , \tag{22}$$

and from (16)

$$P_0 = N\cos\Theta - B\sin\Theta. \tag{23}$$

Since $Q_0 = T \times P_0$, $T \times N = B$, and $T \times B = -N$, it follows from (23) that

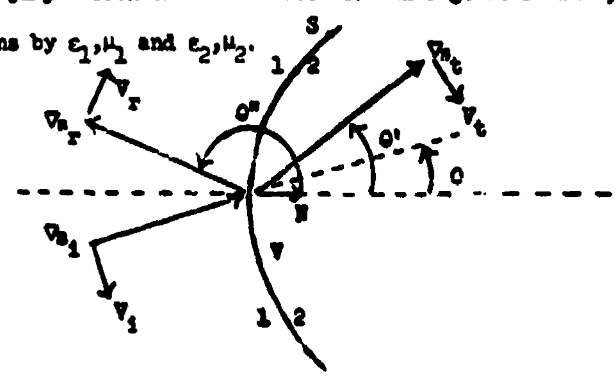
$$Q_0 = N\sin\Theta + B\cos\Theta. \tag{24}$$

(22), (23), and (24) give the rotation of the polarization vectors P_0, Q_0

relative to B and N. If the ray remains in one plane, then $\gamma \equiv 0$, and θ is constant along a ray. A sufficient (but not a necessary) condition for this is that the medium be homogeneous, i.e., that n = constant.

B5 Reflection and transmission at an interface.

In this section, we focus our attention on an interface or surface S which separates two regions in which ε and μ are smooth functions. These functions may have jump discontinuities across S. In regions 1 and 2, we denote the functions by ε_1, μ_1 and ε_2, μ_2.



The values of the reflected and transmitted fields at the interface can be derived from the well-known discontinuity conditions for the electromagnetic field, which require the continuity of the tangential components of $\hat{\mathcal{E}}$ and $\hat{\mathcal{H}}$ and therefore of $\mathcal{E}$ and $\mathcal{H}$. Since we assume that the incident and reflected fields are defined in region 1, and the transmitted field is defined in region 2, the conditions become

$$N \times (\mathcal{E}_i + \mathcal{E}_r) = N \times \mathcal{E}_t, \quad N \times (\mathcal{H}_i + \mathcal{H}_r) = N \times \mathcal{H}_t, \text{ on } S. \tag{1}$$

Here N denotes a unit vector normal to Γ pointing in the direction from region 1 to region 2 (see figure).

Each field is of the form (2.1), so it satisfies (2.2), (2.3), etc. By inserting the expansions (2.1) into (1) we obtain

$$s_i = s_r = s_t, \text{ on } S, \tag{2}$$

and, for the zero-order coefficients,

$$N \times (E_i + E_r) = N \times E_t, \quad N \times (H_i + H_r) = N \times H_t, \text{ on } S. \tag{3}$$

From (2.3), we have

$$\nabla s_i \times H_i + c_0 \varepsilon_1 E_i = 0, \qquad \nabla s_i \times E_i - c_0 \mu_1 H_i = 0, \tag{4}$$

$$\nabla s_r \times H_r + c_0 \varepsilon_1 E_r = 0, \qquad \nabla s_r \times E_r - c_0 \mu_1 H_r = 0, \tag{5}$$

$$\nabla s_t \times H_t + c_0 \varepsilon_2 E_t = 0, \qquad \nabla s_t \times E_t - c_0 \mu_2 H_t = 0. \tag{6}$$

We now introduce the parametric equation for the surface S,

$$X = X(\xi_1, \xi_2) = X(\xi). \tag{7}$$

Then (2) may be written

$$s_i[X(\xi)] = s_r[X(\xi)] = s_t[X(\xi)]. \tag{8}$$

Differentiation of (8) with respect to ξ_1 and ξ_2 yields

$$\nabla s_i \cdot X_{\xi_j} = \nabla s_r \cdot X_{\xi_j} = \nabla s_t \cdot X_{\xi_j}; \quad j = 1, 2. \tag{9}$$

Since the vectors X_{ξ_j} are tangential to S, (9) implies that the differences $\nabla s_i - \nabla s_r$, $\nabla s_i - \nabla s_t$, are normal to S, i.e.,

$$\nabla s_r = \nabla s_i + \gamma_r N, \quad \nabla s_t = \nabla s_i + \gamma_t N. \tag{10}$$

It follows that ∇s_r and ∇s_t (and hence the reflected and transmitted rays) at S lie in the plane of incidence determined by ∇s_i and N. (This plane is the plane of the figure.) Furthermore (10) implies that there exists a unit vector V and a real scalar a such that

$$\nabla s_r \times N = \nabla s_i \times N = \nabla s_t \times N = aV. \tag{11}$$

V is perpendicular to the plane of incidence. (In the figure V points into the paper).

By equating the magnitudes of the vectors in (11), we obtain

$$n_1 \sin \theta'' = n_1 \sin \theta = n_2 \sin \theta' = a. \tag{12}$$

The angles θ, θ', θ'' appear in the figure. It is clear that the only solutions of (12) consistent with the figure are

$$\theta'' = \pi - \theta, \tag{13}$$

and

$$\sin \theta' = \frac{n_1}{n_2} \sin \theta; \quad 0 \leq \theta' \leq \pi/2. \tag{14}$$

(13) and (14) may be recognized as the <u>law of reflection</u>, and <u>Snell's law of refraction</u>. If

$$\frac{n_1}{n_2} \sin \theta > 1 \tag{15}$$

then (14) has no real solution θ'. This is the case of total reflection discussed in Section A13.

In order to determine the amplitudes of the reflected and transmitted fields, it is convenient to introduce the 3 unit vectors

$$V_i = V \times \nabla s_i/n_1, \qquad V_r = V \times \nabla s_r/n_1, \qquad V_t = V \times \nabla s_t/n_2 \tag{16}$$

which appear in the figure. Since E_i is orthogonal to ∇s_i, it can be expressed as a linear combination of V_i and V. The same assertion applies to H_i, E_r, H_r, E_t, H_t. Thus we may set

$$\sqrt{\epsilon_1}E_i = \alpha_i V_i + \beta_i V, \quad \sqrt{\epsilon_1}E_r = \alpha_r V_r + \beta_r V, \quad \sqrt{\epsilon_2}E_t = \alpha_t V_t + \beta_t V. \tag{17}$$

Then it is easy to show that (4-6) are satisfied provided

$$\sqrt{\mu_1}\, H_i = -\beta_i V_i + \alpha_i V, \quad \sqrt{\mu_1}\, H_r = -\beta_r V_r + \alpha_r V, \quad \sqrt{\mu_2}H_t = -\beta_t V_t + \alpha_t V. \tag{18}$$

In order to apply (3) we first observe that $V_i \times N = (V \times \nabla s_i/n_1) \times N = -V(\nabla s_i \cdot N/n_1) = -V \cos\theta$, and therefore

$$\sqrt{\epsilon_1}(E_i \times N) = \alpha_i (V_i \times N) + \beta_i (V \times N) = -\alpha_i V \cos\theta + \beta_i V \times N. \tag{19}$$

Thus

$$E_i \times N = \frac{-\alpha_i}{\sqrt{\epsilon_1}} V \cos\theta + \frac{\beta_i}{\sqrt{\epsilon_1}} V \times N, \quad H_i \times N = \frac{\beta_i}{\sqrt{\mu_1}} V \cos\theta + \frac{\alpha_i}{\sqrt{\mu_1}} V \times N. \tag{20}$$

Similarly,

$$E_r \times N = \frac{\alpha_r}{\sqrt{\epsilon_1}} V \cos\theta + \frac{\beta_r}{\sqrt{\epsilon_1}} V \times N, \quad H_r \times N = -\frac{\beta_r}{\sqrt{\mu_1}} V \cos\theta + \frac{\alpha_r}{\sqrt{\mu_1}} V \times N, \tag{21}$$

$$E_t \times N = \frac{-\alpha_t}{\sqrt{\epsilon_2}} V \cos\theta' + \frac{\beta_t}{\sqrt{\epsilon_2}} V \times N, \quad H_t \times N = \frac{\beta_t}{\sqrt{\mu_2}} V \cos\theta' + \frac{\alpha_t}{\sqrt{\mu_2}} V \times N. \tag{22}$$

Here we have used the fact that

$$\cos\theta'' = \cos(\pi - \theta) = -\cos\theta. \tag{23}$$

If we now insert (20-22) into (3) we obtain

$$\begin{aligned} \alpha_r - \alpha_i &= -\sqrt{\frac{\epsilon_1}{\epsilon_2}}\,\frac{\cos\theta'}{\cos\theta}\,\alpha_t, \quad \alpha_r + \alpha_i = \sqrt{\frac{\mu_1}{\mu_2}}\,\alpha_t, \\ \beta_r - \beta_i &= -\sqrt{\frac{\mu_1}{\mu_2}}\,\frac{\cos\theta'}{\cos\theta}\,\beta_t, \quad \beta_r + \beta_i = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\,\beta_t. \end{aligned} \tag{24}$$

These equations can be solved for α_r, β_r, α_t, β_t in terms of the components α_i, β_i of the incident field. The result is

$$\frac{\alpha_r}{\alpha_i} = \frac{\sqrt{\frac{\mu_1}{\mu_2}} - \sqrt{\frac{\epsilon_1}{\epsilon_2}}\,\frac{\cos\theta'}{\cos\theta}}{\sqrt{\frac{\mu_1}{\mu_2}} + \sqrt{\frac{\epsilon_1}{\epsilon_2}}\,\frac{\cos\theta'}{\cos\theta}}, \qquad \frac{\beta_r}{\beta_i} = \frac{\sqrt{\frac{\epsilon_1}{\epsilon_2}} - \sqrt{\frac{\mu_1}{\mu_2}}\,\frac{\cos\theta'}{\cos\theta}}{\sqrt{\frac{\epsilon_1}{\epsilon_2}} + \sqrt{\frac{\mu_1}{\mu_2}}\,\frac{\cos\theta'}{\cos\theta}} \tag{25-A}$$

$$\frac{\alpha_t}{\alpha_i} = \frac{2}{\sqrt{\frac{\mu_1}{\mu_2}} + \sqrt{\frac{\epsilon_1}{\epsilon_2}}\,\frac{\cos\theta'}{\cos\theta}}\,, \qquad \frac{\beta_t}{\beta_i} = \frac{2}{\sqrt{\frac{\epsilon_1}{\epsilon_2}} + \sqrt{\frac{\mu_1}{\mu_2}}\,\frac{\cos\theta'}{\cos\theta}}\,. \tag{25-B}$$

The components α, β of the fields are, respectively, parallel and normal to the plane of incidence. It is sometimes customary to use the notation for the parallel and normal components of the electric field:

$$E_{ip} = \frac{\alpha_i}{\sqrt{\epsilon_1}}\,,\quad E_{rp} = \frac{\alpha_r}{\sqrt{\epsilon_1}}\,,\quad E_{tp} = \frac{\alpha_t}{\sqrt{\epsilon_2}}\,,\quad E_{in} = \frac{\beta_i}{\sqrt{\epsilon_1}}\,,\quad E_{rn} = \frac{\beta_r}{\sqrt{\epsilon_1}}\,,\quad E_{tn} = \frac{\beta_t}{\sqrt{\epsilon_2}}\,. \tag{26}$$

If we assume that

$$\mu_1 = \mu_2 = \mu \tag{27}$$

and set $n_1 = c_0\sqrt{\epsilon_1\mu}$, $n_2 = c_0\sqrt{\epsilon_2\mu}$, then (25) becomes

$$E_{rp} = E_{ip}\,\frac{1 - \frac{n_1\cos\theta'}{n_2\cos\theta}}{1 + \frac{n_1\cos\theta'}{n_2\cos\theta}}\,, \qquad E_{rn} = E_{in}\,\frac{\frac{n_1}{n_2} - \frac{\cos\theta'}{\cos\theta}}{\frac{n_1}{n_2} + \frac{\cos\theta'}{\cos\theta}}\,,$$

$$E_{tp} = E_{ip}\,\frac{2\frac{n_1}{n_2}}{1 + \frac{n_1\cos\theta'}{n_2\cos\theta}}\,, \qquad E_{tn} = E_{in}\,\frac{2\frac{n_1}{n_2}}{\frac{n_1}{n_2} + \frac{\cos\theta'}{\cos\theta}}\,. \tag{28}$$

These formulas are identical to the Fresnel formulas for reflection and transmission of a plane electromagnetic wave at a plane interface. We have, of course, shown that they are valid for the zero-order terms of the asymptotic expansion of an arbitary electromagnetic wave at an arbitrary (smooth) interface. By using the results (28) as initial conditions for the electric field on the reflected and transmitted rays, the zero-order reflected and transmitted fields can be found away from the interface.

B6 Reflection from a perfectly conducting surface

The well-known condition on the electromagnetic field at the surface of a perfect conductor is that the tangential component of $\hat{\mathcal{E}}$ must vanish. In contrast with section 5, we have only incident and reflected fields, and the boundary condition may be stated in the form

$$N \times (E_i + E_r) = 0. \tag{1}$$

The consequences of this condition can be obtained easily by simply modifying the equations of section 5 in an obvious way. In this section we list the modified equations, using the same equation numbers to facilitate the comparison. Thus we obtain

$$s_i = s_r, \text{ on } S, \tag{2}$$

$$N \times (E_i + E_r) = 0 \tag{3}$$

$$C^* = \kappa = 0 \tag{13}$$

$$\sqrt{\epsilon}\, E_i = \alpha_i V_i + \beta_i V, \qquad \sqrt{\epsilon}\, E_r = \alpha_r V_r + \beta_r V \tag{17}$$

$$\sqrt{\mu}\, H_i = -\beta_i V_i + \alpha_i V, \qquad \sqrt{\mu}\, H_r = -\beta_r V_r + \alpha_r V \tag{18}$$

$$\alpha_r = \alpha_i\,, \qquad \beta_r = -\beta_i, \tag{25}$$

$$E_{rp} = E_{ip}\,, \qquad E_{rn} = -E_{in}. \tag{28}$$

Here, as in section 5, subscripts p and n denote components of E_i and E_r parallel and normal to the plane of incidence. As before, these values can be used as initial conditions to determine the reflected field all along the reflected rays.

§ 7. Radiation from sources, diffraction, summary

In order to discuss radiation from sources and diffraction, only slight modifications of the results of chapter A are required. In this section we present those modifications, together with a summary of the results of the present chapter. Equations which are taken from earlier sections have numbers to their left which indicate their origin.

In M.K.S. units, the real time-harmonic electric and magnetic fields are given by

$$(1.5) \qquad \hat{\mathcal{E}}(X,t) = \mathrm{Re}\left[\mathcal{E}(X)e^{-i\omega t}\right], \quad \hat{\mathcal{H}}(X,t) = \mathrm{Re}\left[\mathcal{H}(X)e^{-i\omega t}\right]. \tag{1}$$

The complex vectors $\mathcal{E}(X)$ and $\mathcal{H}(X)$ satisfy

(1.6) $\nabla \times \mathcal{H} + i\omega\epsilon\,\mathcal{E} = \sigma_1 \mathcal{E}\ , \quad \nabla \times \mathcal{E} - i\omega\mu\,\mathcal{H} = 0.$ (2)

Here $\epsilon(X)$, $\mu(X)$, and $\sigma_1(X)$ are given functions which characterize the medium.

For large $k = \omega/c_o$ ($c_o = 3 \times 10^8$ meter/sec) we introduce the asymptotic expansions

(2.1) $\mathcal{E} \sim e^{iks} \sum_{m=0}^{\infty} (ik)^{-m} E_m, \qquad \mathcal{H} \sim e^{iks} \sum_{m=0}^{\infty} (ik)^{-m} H_m.$ (3)

The zero order amplitude vectors, $E_o = E$ and $H_o = H$ satisfy

(2.3) $\nabla s \times H + c_o \epsilon E = 0\ , \quad \nabla s \times E - c_o\,\mu\, H = 0,$ (4)

and

(2.4) $E \cdot H = E \cdot \nabla s = H \cdot \nabla s = 0,$ (5)

while $s(X)$ satisfies the eiconal equation

(2.6) $(\nabla s)^2 = n^2(X).$ (6)

Here

(2.7) $n^2 = c_o^{\,2}\epsilon\mu = \frac{\epsilon\mu}{\epsilon_o\mu_o} = \frac{c_o^{\,2}}{c^2(X)}\ ; \quad c^2(X) = \frac{1}{\epsilon(X)\mu(X)}\ .$ (7)

Equation (4) shows that H can be obtained from a knowledge of E (and vice-versa). (5) shows that E and H are mutually orthogonal and each orthogonal to ∇s, i.e., to the ray. From (6) it follows that the phase $s(X)$

and rays are given by the equations of sections A2 and A6.

We define the zero-order energy density function

$$(3.5) \qquad v = \frac{1}{2}\epsilon\, E\cdot\bar{E} = \frac{1}{2}\mu\, H\cdot\bar{H}, \qquad (8)$$

and the polarization vectors, P and Q, by

$$(3.34) \qquad E = \sqrt{\frac{2v}{\epsilon}}\, P, \quad H = \sqrt{\frac{2v}{\mu}}\, Q. \qquad (9)$$

Then

$$P\cdot\bar{P} = 1, \quad Q\cdot\bar{Q} = 1, \quad P\cdot Q = 0, \qquad (10)$$

and P and Q each satisfy the first order system of ordinary differential equations

$$(3.36) \qquad \frac{dP}{d\sigma} + \frac{P\cdot\nabla n}{n^2}\nabla s = 0. \qquad (11)$$

Here σ denotes arclength along a ray. The value of v along a ray is given by

$$(3.\ldots) \qquad \frac{v(\sigma)\,\xi(\sigma)}{n(\sigma)} = \frac{v(\sigma_0)\,\xi(\sigma_0)}{n(\sigma_0)} \exp\left\{-\int_{\sigma_0}^{\sigma} \sigma_1\sqrt{\frac{\mu}{\epsilon}}\, d\sigma'\right\}. \qquad (12)$$

Here $\xi(\sigma) = \frac{da(\sigma)}{da(\sigma_1)}$ is the expansion ratio introduced in Section A3. If E and H are given at some point σ_0 on a ray, then at this point v, P, and Q can be obtained from (8) and (9). At any other point σ, on the ray v is given by (12) and P and Q can be obtained by solving (11). Then finally at σ, E and H are given by (9).

The plane of polarization of the electric field is perpendicular to the ray and the ellipticity is constant on a ray. For the special case of linear polarization, i.e., zero ellipticity, additional conclusions can be drawn. In this case,

$$P = aP_0, \quad Q = aQ_0. \qquad (4.3), (4.5) \qquad (13)$$

Here a is a complex number of modulus one and P_0 and Q_0 are real unit vectors which are mutually orthogonal and orthogonal to the ray. Furthermore to zero order

$$\hat{\mathcal{E}} \sim \sqrt{\frac{2\nu}{\epsilon}} \cos[ks - \nu t] P_0, \qquad (4.7) \qquad (14)$$

$$\hat{\mathcal{H}} \sim \sqrt{\frac{2\nu}{\mu}} \cos[ks - \nu t] Q_0. \qquad (4.8) \qquad (15)$$

In addition (for the case of linear polarization) the rotation of the polarization vectors P_0 and Q_0 are given by

$$P_0 = N \cos \Theta - B \sin \Theta, \qquad (4.23) \qquad (16)$$

$$Q_0 = N \sin \Theta + B \cos \Theta. \qquad (4.24) \qquad (17)$$

Here N and B are the unit normal and binormal vectors of the ray, and

$$\Theta = \Theta_0 + \int_{\sigma_0}^{\sigma} \tau \, d\sigma'. \qquad (4.22) \qquad (18)$$

τ is the torsion of the ray.

The conditions for reflection and transmission at an interface are given by (2), (13), (14), and (25) or (28) of section 5. Similarly the conditions for reflection at a perfect conductor are given by (2), (13) and (25) or (28) of section 6.

Homogeneous media

Let us call a medium homogeneous if $n(X)$ = constant. Since in applications μ is almost always a constant, constancy of n means that ϵ is constant too. In this case the rays are straight lines and the phase is given by

$$s = s_o + n\sigma . \qquad (A4.2) \qquad (19)$$

From (11) we see that the vectors P and Q are constant on a ray and hence (to zero order) the direction of the major and minor axes of the ellipse of polarisation are constant. The expansion ratio is given by

$$\xi(\sigma) = \frac{(\rho_1 + \sigma)(\rho_2 + \sigma)}{\rho_1 \rho_2}, \qquad (A4.4) \qquad (20)$$

and hence (12) becomes

$$v(\sigma) = v(\sigma_o) \frac{(\rho_1 + \sigma_o)\,(\rho_2 + \sigma_o)}{(\rho_1 + \sigma)\,(\rho_2 + \sigma)} \exp\left\{ -\sqrt{\frac{\mu}{\epsilon}} \int_{\sigma_o}^{\sigma} \sigma_1 d\sigma' \right\} . \qquad (21)$$

ρ_1 and ρ_2 are constants on a ray. From (9) and (21) we now have

$$E(\sigma) = E(\sigma_o) \left[\frac{(\rho_1 + \sigma_o)(\rho_2 + \sigma_o)}{(\rho_1 + \sigma)\,(\rho_2 + \sigma)}\right]^{1/2} \exp\left\{ -\frac{1}{2}\sqrt{\frac{\mu}{\epsilon}} \int_{\sigma_o}^{\sigma} \sigma_1 \, d\sigma' \right\} \qquad (22)$$

and a similar equation for H.

<u>Radiation from sources</u>

As in section A7, point, line, and surface sources may be characterized by giving the values of s, E, and H (or of s, w, P, and Q) on the source manifold M. This is particularly convenient when M is a secondary source such as occurs in reflection, transmission and diffraction. The results of section A7 can easily be applied here. We first rewrite (12) in the form

$$\frac{v(\sigma)}{n(\sigma)} = \frac{v(\sigma_o)}{n(\sigma_o)} \frac{da(\sigma_o)}{da(\sigma)} \lambda; \qquad \lambda = \exp\left\{ -\int_{\sigma_o}^{\sigma} \sigma_1 \sqrt{\frac{\mu}{\epsilon}}\, d\sigma' \right\}. \tag{23}$$

If M is a (non-characteristic) surface S, then on every outgoing ray we may measure σ from S and $v(\sigma)$ is given by (23) with σ_o replaced by 0.

If M is a point, let $d\Omega$ be an element of solid angle of the starting directions of the rays. Then for $\sigma_o \to 0$ $da(\sigma_o) \sim \sigma_o^2 d\,\Omega$ and

$$\frac{v(\sigma)}{n(\sigma)} = \frac{\tilde{v}(0)}{n(0)} \frac{d\,\Omega}{da(\sigma)} \lambda_o; \qquad \lambda_o = \exp\left\{ -\int_{0}^{\sigma} \sigma_1 \sqrt{\frac{\mu}{\epsilon}}\, d\sigma' \right\}; \tag{24}$$

where

$$\tilde{v}(0) = \lim_{\sigma \to 0} \sigma^2 v(\sigma). \tag{25}$$

For a homogeneous medium, $da(\sigma) = \sigma^2 d\Omega$. Hence

$$v(\sigma) = \frac{\tilde{v}(0)}{\sigma^2} \lambda_o \qquad \text{or} \qquad E(\sigma) = \frac{\tilde{E}(0)}{\sigma} \lambda_o^{1/2}. \tag{26}$$

Here

$$\tilde{E}(0) = \lim_{\sigma \to 0} \sigma E(\sigma). \tag{27}$$

If M is a curve $da(\sigma_o) \sim d\eta \sin\beta\ \sigma_o d\Theta$. (see (A7.5).) Hence

$$\frac{w(\sigma)}{n(\sigma)} = \frac{\tilde{w}(0)}{n(0)} \quad \frac{d\eta d\Theta \sin\beta}{da(\sigma)} \lambda_o \tag{28}$$

where

$$\tilde{w}(0) = \lim_{\sigma\to 0} \sigma w(\sigma). \tag{29}$$

For a homogeneous medium, $da(\sigma) = \sigma(1 + \frac{\sigma}{\rho_1})\, d\eta d\Theta \sin\beta$. Hence

$$w(\sigma) = \frac{\tilde{w}(0)}{\sigma(1+\sigma/\rho_1)} \lambda_o \text{ or } E(\sigma) = \tilde{E}(0) \left[\frac{\lambda_o}{\sigma(1+\sigma/\rho_1)}\right]^{1/2}. \tag{30}$$

Here

$$\tilde{E}(0) = \lim_{\sigma\to 0} \sigma^{1/2} E(\sigma), \tag{31}$$

and ρ_1 is given by (A7.17).

If M is a characteristic surface S, we set $da(\sigma_o) \sim \sigma_o d\tilde{a}(0)$. Hence

$$\frac{w(\sigma)}{n(\sigma)} = \frac{\tilde{w}(0)}{n(0)} \frac{d\tilde{a}(0)}{da(\sigma)} \lambda_o, \tag{32}$$

where

$$\tilde{w}(0) = \lim_{\sigma\to 0} \sigma w(\sigma). \tag{33}$$

For a homogeneous medium, $da(\sigma) = \sigma(\rho_2 + \sigma) \sin\gamma\, d\Theta_1\, d\Theta_2$, and $d\tilde{a}(0) = \rho_2 \sin\gamma\, d\Theta_1\, d\Theta_2$. Hence

$$w(\sigma) = \tilde{w}(0) \frac{\lambda_o}{\sigma(1+\sigma/\rho_2)} \quad \text{or} \quad E(\sigma) = \tilde{E}(0) \left[\frac{\lambda_o}{\sigma(1+\sigma/\rho_2)}\right]^{1/2}. \tag{34}$$

Here $\tilde{E}(0)$ is given by (31).

Diffraction by edges and vertices

As in section A14 if an electromagnetic wave (3) is incident upon an edge or vertex M, that manifold acts as a secondary source producing a diffracted wave. The phases of incident and diffracted waves satisfy

$$s^d = s^i \quad \text{on } M, \tag{35}$$

$$\tilde{E}^d = (d)E^i \quad \text{on } M. \tag{36}$$

The diffraction coefficient (d) is a matrix. As in section A14, (35) implies the law of edge diffraction. For a homogeneous medium the field diffracted by a vertex or edge is given by (26) or (30), $\tilde{E}(0)$ being given by (36).

For an inhomogeneous medium we shall discuss diffraction by an edge. The discussion for a vertex is similar. We first use (29), (8), and (31) to obtain

$$\tilde{w}^d(0) = \lim_{\sigma \to 0} \sigma w^d(\sigma) = \lim_{\sigma \to 0} \tfrac{1}{2}\sigma\epsilon\, E^d \cdot \bar{E}^d = \tfrac{1}{2}\epsilon(0)\, \tilde{E}^d(0) \cdot \overline{\tilde{E}^d(0)}. \tag{37}$$

Then from (9) we obtain

$$P^d(0) = \lim_{\sigma \to 0} \sqrt{\frac{\epsilon}{2w^d}}\; E^d = \sqrt{\frac{\epsilon(0)}{2\tilde{w}^d(0)}}\; \tilde{E}^d(0). \tag{38}$$

Now on the diffracted ray $w^d(\sigma)$ is given by (28), (37), and (36), while $P^d(\sigma)$ is determined by the system of differential equations (11) and the initial condition (38). Having determined $w^d(\sigma)$ and $F^d(\sigma)$, $E^d(\sigma)$ is given by

$$E^d(\sigma) = \sqrt{\frac{\mu w^d(\sigma)}{\epsilon(\sigma)}}\; F^d(\sigma). \tag{39}$$

The phase of the diffracted field is

$$s^d(\sigma) = s^i(0) + \int_0^{\sigma} n\,d\sigma, \tag{40}$$

and the field associated with the edge diffracted ray is given by

$$\mathcal{E}^d(\sigma) \sim e^{iks^d(\sigma)}\, E^d(\sigma). \tag{41}$$

For an edge of a perfectly conducting thin screen, the diffraction coefficient matrix is given by equation (A12) of [20].

<u>Diffraction by a smooth object in a homogeneous medium.</u>*

The description of the phase functions s^c and s^d and the rays for both the surface wave and the diffracted wave is identical to that given in chapter A. In order to describe the amplitude vectors on a surface ray, we first introduce the vectors D_1, the unit tangent to the ray, D_2 the outward unit normal to the surface, and $D_3 = D_1 \times D_2$. Since the medium is homogeneous the diffracted ray is a surface geodesic, D_2 lies along the unit normal to the ray, and D_3 lies along the binormal. At a point P on the straight diffracted ray which leaves the surface at P_1 we define the triad of vectors

* The following discussion is adapted from section 5 of [29]. The analogous theory for an inhomogeneous medium has not yet been developed.

by setting $D_\nu(P) = D_\nu(P_1)$. We denote the components of the electric field vector $\mathcal{E} \sim e^{iks}E$ in the three directions D_ν by $\mathcal{E}_\nu$.

We now assume that $\mathcal{E}_1$ is zero and that the components $\mathcal{E}_2$ and $\mathcal{E}_3$ propagate independently of each other and satisfy the equations of Chapter A. Then from (A18.13)

$$\mathcal{E}^d_\nu(P) \sim \mathcal{E}^i_\nu(Q_1) \exp\left\{ ik[\bar{n}\tau + n\sigma] \right\} \left[\frac{d\psi(Q_1)}{d\psi(P_1)} \frac{\rho_2}{\sigma(\rho_2+\sigma)}\right]^{1/2}$$

$$\times \sum_j d_{j\nu}(P_1) d_{j\nu}(Q_1) \exp\left\{ -\int_{Q_1}^{P_1} \alpha_{j\nu}(t)\,dt \right\} \;;\quad \nu = 2,3. \qquad (42)$$

Here $\mathcal{E}^i_\nu(Q_1) = \mathcal{E}^i(Q_1)\cdot D_\nu(Q_1)$ is the component of the incident field $\mathcal{E}^i$ at Q_1 in the direction D_ν, and the other quantities are defined in Section A18. The diffraction coefficients $d_{j\nu}$ and the decay exponents $\alpha_{j\nu}$ are different for the two components. At P, the diffracted field associated with a diffracted ray is given by (42) and

$$\mathcal{E}^d(P) = \mathcal{E}^d_2(P)D_2(P) + \mathcal{E}^d_3(P)D_3(P). \qquad (43)$$

For a perfectly conducting smooth object the coefficients d_{j3} and α_{j3} for the tangential component $\mathcal{E}_3$ are the same as those for a scalar field satisfying the condition $u = 0$ on the surface and hence can be obtained from (A19.13 - 15) by setting $z = \infty$. Similarly the coefficients d_{j2} and α_{j2} for the normal component $\mathcal{E}_2$ are the same as those for a scalar field satisfying the condition $\frac{\partial u}{\partial \nu} = 0$ on the surface and hence can be obtained from the same equations by setting $z = 0$. Of course we must also set $\kappa(\lambda) = 0$ and $n(X) = n$ in these equations since the medium is homogeneous.

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